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ZISŤOVANIE POZDĹŽNYCH SÍL NA OHÝBANOM NOSNÍKU

INDEPENDENT FORMATION OF LONGITUDINAL FORCES IN BENT BEAMS

Príspevok sa zaoberá obmedzením deformácií v horizontálnom smere na ohýbaných trámoch. Nastáva pri pootočení koncových prierezov trámu obmedzovaných uložením v podperách. Výsledkom sú prídavné tlakové namáhania. Výsledné vzťahy sú odvodené pre dva zaťažovacie stavy: rovnomerné zaťaženie a osamelé bremená v strede rozpätia trámu. Umožňujú určiť namáhanie a stanoviť účinky na deformácie, ako aj únosnosť pomocou lineárnej, aj nelineárnej teórie. Numerické riešenie ukazuje priaznivý dopad popisovaného fenoménu na pôsobenie trámu. Ukazuje sa, že pri vzraste horizontálnej tuhosti podpier deformácie významne klesajú a napätia vzrastajú. Mimoriadny nárast únosnosti sa konštatuje pri materiáloch s nízkou ťahovou pevnosťou a vysokou tlakovou únosnosťou.

This paper deals with the phenomenon of horizontal constraints being strutted by bent beams. This phenomenon takes place when rotating ends of beam sections encounter the resistance of horizontal constraints. Consequently, reactions occur between the constraints and the beam horizontal which induce a longitudinal compression force. Equations for two load cases: a uniformly distributed and concentrated force applied to the span centre, allowing to determine the value of this force and its influence on deflection and load capacity on the on the basis of the linear and non-linear theory were derived. The analysis carried out on this base showed an advantageous effect of the discussed phenomenon on the beam behaviour. It was proved that as horizontal supports stiffness increases the deflection decreases considerably and its strength increases. An especially significant increase in the load capacity was observed in the case of materials with low tensile strength and high compression strength.

Introduction

In order to carry out static calculations indispensable for the designing of engineering structures designers use theoretical construction models. Every model, especially one of an engineering structure, should reflect the real object as faithfully as possible. Only when the theoretical solution obtained from an analysis of the model conforms to the real object the designer will have the proper basis for designing a structure capable of carrying the loads acting on it and be able to assess its safety.

The models of structure calculations most often contain several simplifications. Although the use of such imperfect models makes calculations easier, the results thus obtained include anticipated errors. They may be, for instance, a lowering of the value of internal forces occuring in the structure (e.g., the first order theory based on the stiffening principle which takes no account of the displacement effect on the value of internal forces or disregards advantageous phenomena occurring in the structure). Generally speaking, the use of a simplified model might result in a condition where the structural safety is endangered or the structure has an excessive (superfluous) capacity reserve (it is uneconomically designed). Neither of these two situations is acceptable.

The correct (tallying with reality) picture of the distribution of internal forces in the structure, a picture issuing from a static analysis is indispensable for a correct designing of an engineering structure. This should be warranted by employed methods of construction analyses. As the bent elements is regarding, solutions conforming to structural mechanics sometimes fail to meet this condition. It was proved [1], [2] that in certain conditions of fixing - due to the strutting of supports - in structure elements such as beams or slabs occurs an intrinsic longitudinal compressing force inconsistent with solutions deduced from structural mechanics which did not allow for such forces. This kind of an incomplete set of internal forces occurring in the structure, though involving an error, has been used most frequently in dimensioning beams and slabs. This need not involve dangerous situations because – as will be shown – the longitudinal compression force exerts an advantageous effect on the work of the bent beam or slab causing an increase in capacity and a decrease in deflections. An analytical solution of this problem is necessary if we are to consider the effect of support strutting in static-endurance calculations.

This paper presents a solution based on certain simplified assumptions concerning only two load schemes: one, uniformly distributed and another, concentrated in the centre of the beam span.

Strutting of supports

The mechanism of the longitudinal force occurring in a free-supported bent beam is explained in Fig. 1. Affected by a transverse load the extreme cross-section of the beam tends to turn.

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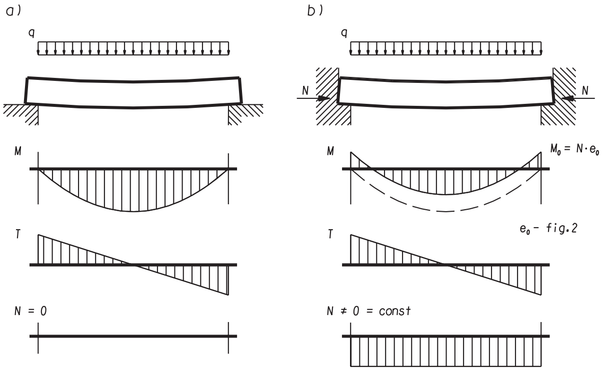

Fig. 1 Differences in internal forces distribution in beam with flexible and stiff support

Due to this rotation the lower edges of these sections shift horizontally (outwards). Depending on their design, beam supports may give way to such displacements or counteract them. The method of support, shown in Fig. 1, does not limit the rotation of the support section. Thus, the "traditional" solution of the beam, compatible with structural mechanics, renders a correct image of the distribution of internal force within the element - bending moments and transverse forces appear there. However, if the supports exhibit a non-zero rigidity in the horizontal direction and are designed so that they are pressured by the lower edges of the rotating support section, there will - as a result of the beam and its supports - occur horizontal forces compressing the beam (Fig. 1b). Besides the bending moments and transverse forces a longitudinal compression force will appear in the beam acting as a self-compression. Moreover, the distribution of the bending moments in a horizontally supported beam tends to change. This takes places because the horizontal force is applied to the beam on an eccentricity e_o (Fig. 2) and thus exerts the moment $M_o = N\, e_o$ on the

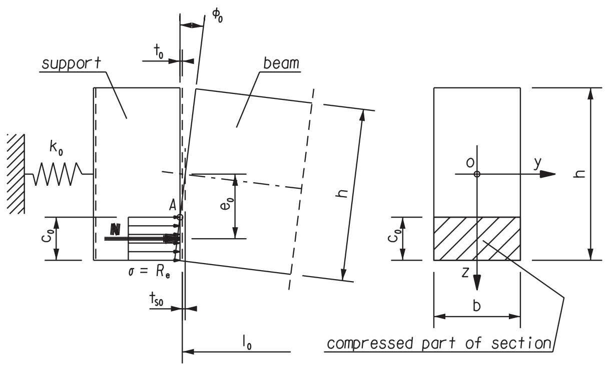

Fig. 2 Contact of beam with support

support generating, in consequence, a reduction of the bending moments in the span. This is illustrated in Fig. 1. The general scheme of support formation shown in Fig. 1b is an indispensable condition for the emergence of the longitudinal force in the beam.

Let us assume that the supports (designed as shown in Fig. 1b) are non- deformed, i. e., their edges remain straight (vertical), yet affected by the strutting force can move elastically in the horizontal direction. Let us also assume that the beam is made of an ideally elastic-plastic material (Fig. 3), which is characterised by an elasticity modulus E and by the plastic boundary R_e. Due to the rotation of the support section caused by an external load

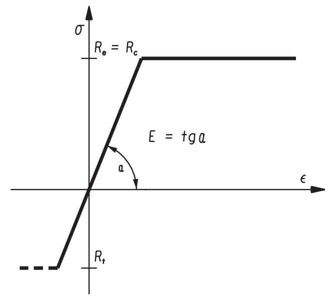

Fig. 3 Material model accepted to analysis

a zone contact appears between the beam and the support (below point A, Fig. 2) where compression stresses occur and a break-off zone (above point A, Fig. 2) without stresses. Thus, according to the accepted assumption, the support edge remains straight and the condition of agreement between beam end displacements (in the compressed zone) and the support must be fulfilled in the entire compression zone. Accordingly:

$$\int_0^l \varepsilon_x\, dx = \Sigma t = const. \qquad \text{for} \qquad z \geq \frac{h}{2} - c_0 .$$

This condition is not fulfilled if we have a free supported working beam in the elastic range of deformations:

$$\int_0^l \varepsilon_x\, dx = \frac{z}{E \cdot J_y} \cdot \int M(x) dx \neq const. \quad \text{for} \quad -\frac{h}{2} \leq z \leq \frac{h}{2} .$$

According to the theory of flat sections the horizontal strains in the extreme section plane vary linearly along the entire height h. Accordingly, the condition of strain agreement, forced by assumptions, will be fulfilled only when plastic deformations appear along the entire height of the compressed zone c_0, in the support-adjacent section. These deformations are accompanied by stresses: $\sigma = R_e$ (Fig. 2). Assuming such uniform stresses, distribution in the support section the longitudinal force will have the value:

$$N = R_e \cdot b \cdot c_0 \tag{1}$$

The unknown value of the range of the compressed zone for section c_0 can be determined on the basis of the geometric dependencies occurring at the beam-support contact (Fig. 2).

This yields:

$$c_0 = \frac{h}{2} - \frac{t_0 + t_{s0}}{\varphi_0} \tag{2}$$

Geometrical values analogous to those in Fig. 2 but relating to the right support of the beam are denoted with index l instead of index 0.

If the support is linearly-elastically deformed and dispays a stiffness characteristic of k_0, its horizontal displacement will be:

$$t_0 = \frac{N}{k_0} = \frac{R_e \cdot b \cdot c_0}{k_0} \qquad (3)$$

The total longitudinal deformation of the beam will be:

$$\Delta l = \frac{N \cdot l}{E \cdot A} = \frac{R_e \cdot c_0 \cdot l}{E \cdot h} \qquad (4)$$

and equals the sum of the longitudinal displacement of the gravity centres of the two support sections:

$$\Delta l = t_{s0} + t_{sl} \qquad (5)$$

The forces equilibrium condition for the forces acting on element ($\Sigma X = 0$) indicates that: $c_0 = c_l$ (for a beam with a constant cross-section). Hence, (2):

$$\frac{t_0 + t_{s0}}{\varphi_0} = \frac{t_l + t_{sl}}{\varphi_l}$$

By substituting: $t_{sl} = \Delta l - t_{s0}$ from (5), we obtain:

$$t_{s0} \cdot \left(1 + \frac{\varphi_0}{\varphi_l}\right) = \frac{\varphi_0}{\varphi_l} \cdot (\Delta l + t_l) - t_0 \qquad (6)$$

If the load acting on the beam is symmetrical, then: $\frac{\varphi_0}{\varphi_l} = 1$ and:

$$t_{s0} = \frac{1}{2} \cdot (\Delta l + t_l - t_0) \qquad (6a)$$

Should, apart from this, the stiffness of the two supports be identical ($k_0 = k_l$) then: $t_0 = t_l$ and the displacement t_{s0} is:

$$t_{s0} = \frac{\Delta l}{2} = \frac{R_e \cdot c_0 \cdot l}{2 \cdot E \cdot h} \qquad (6b)$$

After substituting (3) and (6b) into equation (2) and after the transformations, we get:

$$c_0 = \frac{h}{2 + \frac{R_e}{\varphi_0} \cdot \left(\frac{2 \cdot b}{k_0} + \frac{l}{E \cdot h}\right)} \qquad (7)$$

On the other hand, after substituting dependencies (3) and (6a) into equation (2), we get a more general form of the above quotation, also including the case in which the two beam supports are of different stiffness ($k_0 \neq k_l$):

$$c_0 = \frac{h}{2 + \frac{R_e}{\varphi_0} \cdot \left[b \cdot \left(\frac{1}{k_0} + \frac{1}{k_l}\right) + \frac{l}{E \cdot h}\right]} \qquad (7a)$$

Denoting: $\frac{1}{k'} = \frac{1}{k_0} + \frac{1}{k_l}$, we get:

$$c_0 = \frac{h}{2 + \frac{R_e}{\varphi_0} \cdot \left[\frac{b}{k'} + \frac{l}{E \cdot h}\right]} \qquad (7b)$$

It is evident that the value denoted by k' is the substitute stiffness of the system of the two horizontal support constraints, which are connected in a row. When calculating the range of the compressed zone in support section of the beam, it is more convenient to use the stiffness of the entire frame system, i. e., formula (7b), considered to be more universal.

Expressions (7, 7a, 7b) show that in order to determine the range of the compressed zone in the support section, it is necessary to calculate the compression force value (1). In addition, we must have the value of the rotation angle of this section. Apart from the section stiffness and span, its value is also affected by the value and method of beam loading. We may say that φ_0 is a specific feature of a given load regime.

Subsequent considerations will be confined to two specific load cases: uniformly distributed along the entire beam length and concentrated at the span centre. These load regimes were chosen due to their generality. What is more, they are symmetrical and previously introduced equations (7) and (7b) may, therefore, be used. The solution of such (simple) methods of beam-loading will adequately show the advantages of a beam-end horizontal support.

Uniform load

Figure 4 illustrates a static diagram and loads acting on the beam in question. Apart from the transverse load q, we have an axial force N acting on the beam, resulting from the strutting of support and supports moments: $M_0 = N \cdot e_0$, resulting from the eccentric action of the compression force N. Let us assume that

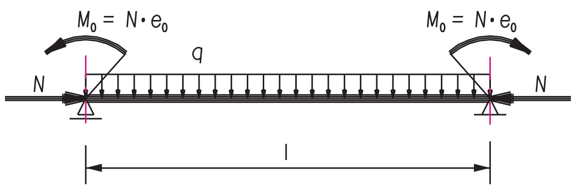

Fig. 4 Internal forces acted on beam

the beam is elastically deformed and the local plasticization of the material in the support-adjacent zones has little effect on the deformation of the element. If we apply the linear theory, the solution is trivial: the rotation of the support section is a superposition of the rotations produced by the transverse load q and the support moment M_0:

$$\varphi_0 = \frac{1}{E \cdot J_y} \cdot \left(\frac{1}{24} \cdot gl^3 - \frac{1}{2} \cdot N \cdot e_0 \cdot l\right) \qquad (8)$$

After substituting dependence (1) into the above quotation, and then substituting it to formula (7b), for a rectangular beam $b \cdot h$ in size, we get the following third degree equation:

$$c_0^3 - \frac{3}{2} \cdot h \cdot c_0^2 + \left(\frac{1}{6} \cdot \frac{q \cdot l^2}{R_e \cdot b} + \frac{2}{3} \cdot h^2 + \frac{1}{6} \cdot \frac{E \cdot b \cdot h^3}{l \cdot k'} \right) \cdot$$

$$\cdot c_0 - \frac{q \cdot h \cdot l^2}{12 \cdot R_e \cdot b} = 0 \qquad (9)$$

Testing its discriminant, we can say that it has always two complex roots and one real root which is the sought value of the compressed zone range for section c_0 at the support. It is also possible to derive an algebraic expression to determine the real root of this equation. However, its form is complex and difficult to analyse. This is why it is much easier to solve the above equation using a computer.

Equation (9) was derived in accordance with the linear theory, based on the stiffening principle. Hence, it fails to take into account the effect of deflection on the distribution of bending moments in the beam and, thereby, on the value of the rotation angle of the support section. This is not without importance in a situation where the element is affected by a compression force. The non-linear theory will assist in obtaining the accurate solution.

According to the 2nd order theory, the bending moments in the beam under discussion assume the following distribution:

$$M(x) = M_q(x) - N \cdot [e_0 - w(x)] = \\ = M_q(x) - N \cdot e_0 + N \cdot w(x) \qquad (10)$$

The total beam deflection $w(x)$ is the sum of the deflection due to the transverse load $w_q(x)$ and the deflection due to the longitudinal force $w_N(x)$ (at a concurrent appearance of load q):

$$w(x) = w_q(x) + w_N(x) \qquad (11)$$

By substituting (10) into the equation, we get:

$$M(x) = M_q(x) + N \cdot w_q(x) + N \cdot w_N(x) - N \cdot e_0. \qquad (12)$$

The unknown function $w_N(x)$ expressing a deflection increase due to the longitudinal force N, will be determined by means of the differential equation:

$$M_N(x) = - EJ_y \cdot w''_N(x) = - N \cdot [e_0 - w_q(x) - w_N(x)] \qquad (13)$$

After the transformation, we get:

$$EJ_y \cdot w''_N(x) + N \cdot w_N(x) = - N \cdot w_q(x) + N \cdot e_0 \qquad (14)$$

As mentioned above, the deflection wq(x) results from the transverse load acting on the beam (it is the axis equation of a deflected beam under this load only). As we know, this equation has - for a uniformly loaded beam - the following form:

$$w_q(x) = \frac{q}{24EJ_y} \cdot x^4 - \frac{q \cdot l}{12EJ_y} \cdot x^3 + \frac{ql^3}{24EJ_y} \cdot x \qquad (15)$$

whereas the bending moments equation:

$$M_q(x) = - \frac{q}{2} \cdot x^2 + \frac{q \cdot l}{2} \cdot x \qquad (16)$$

After substituting dependence (15) into the differential equation (14) and solving it, we get an equation for the deflection at any point of the beam, the deflection due to the longitudinal force N, at the concurrent uniform load q:

$$w_N(x) = - \frac{q}{24EJ_y} x^4 + \frac{q \cdot l}{12EJ_y} x^3 + \frac{q}{2N} x^2 - \left(\frac{q \cdot l^3}{24EJ_y} + \right.$$

$$\left. + \frac{q \cdot l}{2N} \right) x + e_0 - \frac{q \cdot EJ_y}{N^2} + \left(\frac{q \cdot EJ_y}{N^2} - e_0 \right) \cdot \cos(\beta \cdot x) +$$

$$+ \left[\frac{q \cdot EJ_y}{N^2} \cdot \left(\frac{1 - \cos(\beta \cdot l)}{\sin(\beta \cdot l)} \right) + e_0 \cdot \left(\frac{\cos(\beta \cdot l) - 1}{\sin(\beta \cdot l)} \right) \right] \cdot$$

$$\cdot \sin(\beta \cdot x) \qquad (17)$$

The dependence thus obtained can be substituted into formula (12) together with the equations (15) and (16). After its transformation, we get the equation of moments in a beam bent by the uniform load q and compressed eccentrically by force N:

$$M(x) = \frac{q \cdot EJ_y}{N} \left[\cos(\beta \cdot x) - 1 + \left(\frac{1 - \cos(\beta \cdot l)}{\sin(\beta \cdot l)} \right) \cdot \right.$$

$$\left. \cdot \sin(\beta \cdot x) \right] + N \cdot e_0 \cdot \left[\left(\frac{\cos(\beta \cdot l) - 1}{\sin(\beta \cdot l)} \right) \cdot \sin(\beta \cdot x) - \right.$$

$$\left. - \cos(\beta \cdot x) \right]$$

$$\text{for } N \neq 0 \qquad (18)$$

Starting from the differential equation of the deflected beam axis, we can show that, for such a distribution of bending moments in the beam, the rotation angle of the support section would be:

$$\varphi_0 = \frac{1}{\beta} \cdot \left(\frac{N \cdot e_0}{EJ_y} - \frac{q}{N} \right) \cdot \left(\frac{\cos(\beta \cdot l) - 1}{\sin(\beta \cdot l)} \right) - \frac{q \cdot l}{2N}$$

$$\text{for } N \neq 0 \qquad (19)$$

After substituting dependencies (19) and (1) into equation (7b), we shall get (for a rectangular section) the equation:

In this equation, the sought value of the compressed zone range for support section c_0 occurs in an implicit form. It is, therefore, impossible to solve it with general numbers. Equation (20) can be solved only numerically. Knowing the range of c_0 equation (1) will yield the axial compression force, whereas equation (18) - the bending moment at any section of the beam.

On the other hand, the axis equation of the deformed beam may be expressed:

$$(2 \cdot c_0 - h) \cdot \left\{ \frac{E \cdot h^3}{12 \cdot R_e \cdot c_0} \cdot \left[\frac{6 \cdot R_e \cdot c_0}{E \cdot h^3} \cdot (h - c_0) - \frac{q}{R_e \cdot b \cdot c_0} \right] \cdot \left[\frac{\cos\left(\sqrt{\frac{12 \cdot R_e \cdot c_0}{E \cdot h^3}} \cdot l\right) - 1}{\sin\left(\sqrt{\frac{12 \cdot R_e \cdot c_0}{E \cdot h^3}} \cdot l\right)} \right] - \left(\frac{q \cdot l}{2 \cdot R_e \cdot b \cdot c_0} \right) \right\} +$$

$$+ R_e \cdot c_0 \cdot \left(\frac{b}{k'} + \frac{l}{E \cdot h} \right) = 0 \qquad \text{for} \qquad k' \neq 0 \tag{20}$$

$$w(x) = \frac{q \cdot E \cdot J_y}{N^2} \left[\cos(\beta \cdot x) + \frac{N}{2 \cdot E \cdot J_y} \cdot x^2 + \left(\frac{1 - \cos(\beta \cdot l)}{\sin(\beta \cdot l)} \right) \cdot \sin(\beta \cdot x) - 1 \right] + \frac{1}{2} \cdot (h - c_0) \cdot \left[\left(\frac{\cos(\beta \cdot l) - 1}{\sin(\beta \cdot l)} \right) \cdot \right.$$

$$\left. \cdot \sin(\beta \cdot x) - \cos(\beta \cdot x) + 1 \right] - \frac{q \cdot l}{2 \cdot N} \cdot x \qquad \text{for} \qquad N \neq 0 \tag{21}$$

Concentrated force load

For a beam loaded at its span centre with a concentrated force P we can derive the equation for bending moments, the deflection line and the equation to determine the range of the compressed zone in the support section in a manner analogous to a uniformly loaded beam. For this reason, we confine ourselves to the solution only.

According to the linear theory, it will be:

$$\varphi_0 = \frac{1}{EJ_y} \left(\frac{1}{16} \cdot P \cdot l^2 - \frac{1}{2} \cdot N \cdot e_0 \cdot l \right) \tag{8a}$$

$$c_0^3 - \frac{3}{2} \cdot h \cdot c_0^2 + \left(\frac{P \cdot l}{4 \cdot R_e \cdot b} + \frac{2}{3} \cdot h^2 + \frac{P \cdot l \cdot h}{6 \cdot k' \cdot l} \right) \cdot c_0 - $$

$$- \frac{P \cdot l \cdot h}{16 \cdot R_e \cdot b} = 0 \tag{9a}$$

According to the 2nd order theory:

$$m(x) = \frac{P}{2 \cdot \beta} \cdot \frac{1}{\cos\left(\frac{\beta \cdot l}{2}\right)} \cdot \sin(\beta \cdot x) - N \cdot e_0 \cdot$$

$$\cdot \left(tg\left(\frac{\beta \cdot l}{2}\right) \cdot \sin(\beta \cdot x) + \cos(\beta \cdot x) \right) \tag{18a}$$

for $x \leq \frac{1}{2}$

$$\varphi_0 = \frac{P}{2 \cdot N} \left(\frac{1}{\cos\left(\frac{\beta \cdot l}{2}\right)} - 1 \right) - \beta \cdot e_0 \cdot tg\left(\frac{\beta \cdot l}{2}\right) \tag{19a}$$

Equations (8a), (9a), (18a), (19a), (20a) and (21a) correspond to the analogous formulas derived for a uniformly loaded beam (8), (9), (18), (19), (20) and (21) respectively.

$$(2 \cdot c_0 - h) \cdot \left\{ \frac{P}{2 \cdot R_e \cdot b \cdot c_0} \cdot \left[\frac{1}{\cos\left(\sqrt{\frac{3 \cdot R_e \cdot c_0}{E \cdot h^3}} \cdot l\right)} - 1 \right] - (h - c_0) \cdot \sqrt{\frac{3 \cdot R_e \cdot c_0}{E \cdot h^3}} \cdot tg\left(\frac{3 \cdot R_e \cdot c_0}{E \cdot h^3} \cdot l \right) \right\} +$$

$$+ R_e \cdot c_0 \cdot \left(\frac{b}{k'} + \frac{l}{E \cdot h} \right) = 0 \qquad \text{for} \qquad k' \neq 0 \tag{20a}$$

$$w(x) = \left(\frac{P}{2 \cdot \beta \cdot l} \cdot \frac{1}{\cos\left(\frac{\beta \cdot l}{2}\right)} - \frac{1}{2} \cdot (h - c_0) \cdot tg\left(\frac{\beta \cdot l}{2}\right) \right) \cdot \sin(\beta \cdot x) - \frac{1}{2} \cdot (h - c_0) \cdot \cos(\beta \cdot x) - \frac{P}{2 \cdot N} \cdot x + \frac{1}{2} \cdot (h - c_0)$$

$$\text{for } x \leq \frac{1}{2} \tag{21a}$$

Using the equations derived for either of the two load cases we can determine the distribution of the internal forces in the beam. The range of the compressed zone on the support is determined for a given beam geometry (l, b, h) support stiffness (k'), material characteristics (E, R_e) and load (q or P). For the linear theory we use formulas (9) or (9a) and (20) or (20a) - for the non-linear theory, respectively. Knowing the compressed zone range, we can calculate from formula (1) the value of the longitudinal compression force and then determine the distribution of the bending moments (for the non-linear theory, according to formula (18) or (18a). The transverse force distribution is the same as for a beam bent only. Equation (21) or (21a) may yield a deflection at any point of the beam.

Analysis of results

The derived equations show how many factors affect, in a general manner, the behaviour of a beam supported at its ends by horizontal constraints. It is essential on the basis of which theory, linear or non-linear, these calculations are carried out. Of all factors included in our considerations the most important role is that of the support stiffness k'. It is also interesting to know the way it affects the distribution of the internal forces, deflection and, above all, the load capacity of the beam.

Answers to these questions for the two cases here discussed are given by the following graphs - made for an exemplary beam:

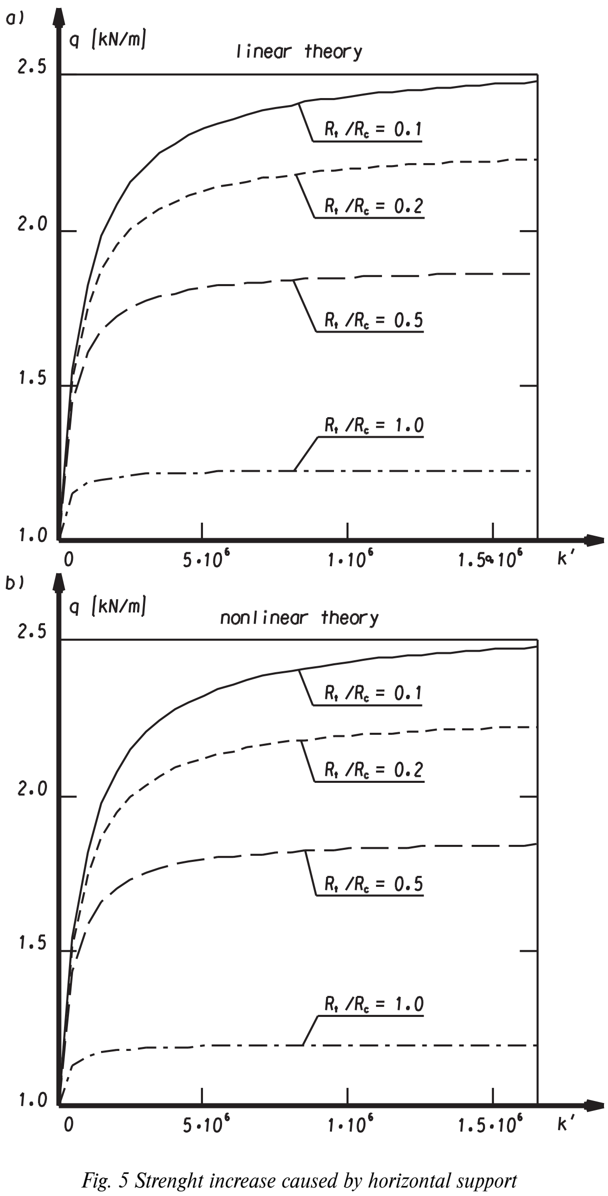

Fig. 5 Strenght increase caused by horizontal support constraints by uniform loading

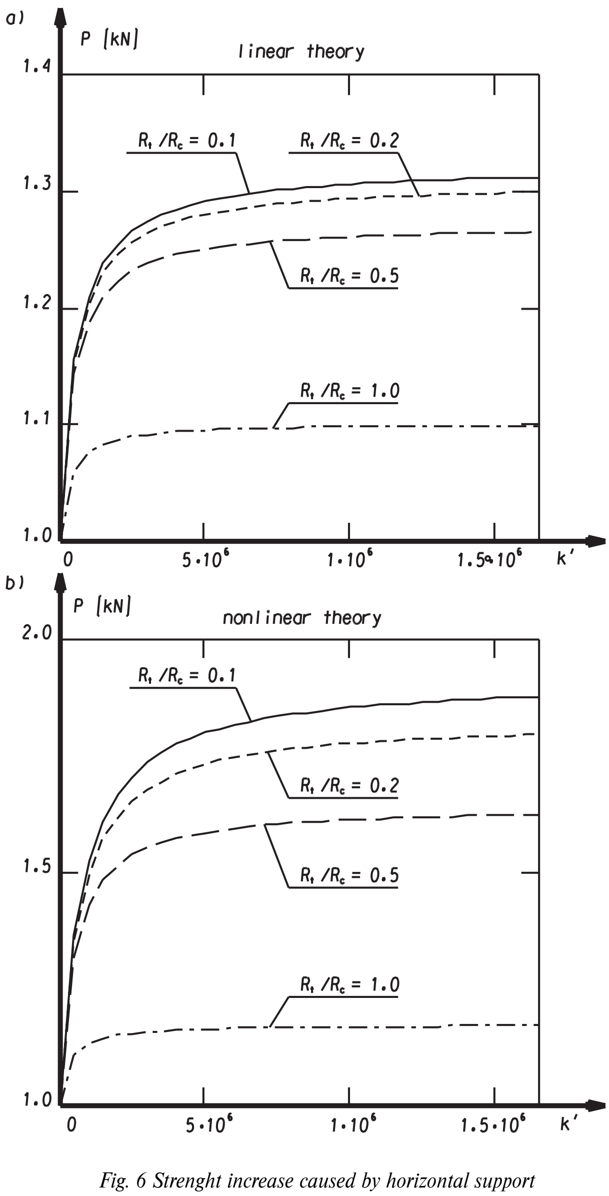

Fig. 6 Strenght increase caused by horizontal support constraints by concentrated loading

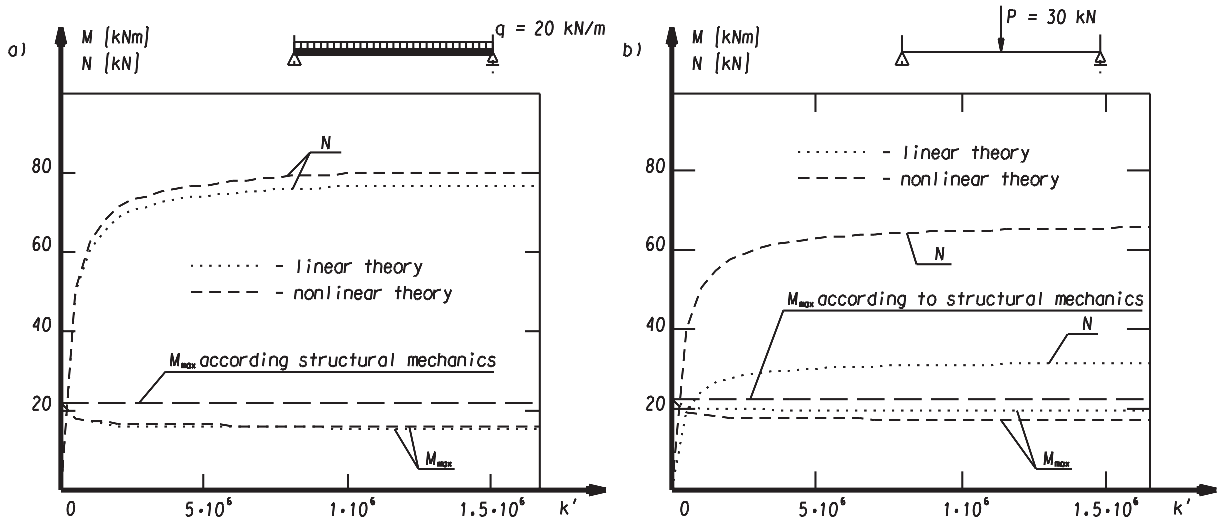

Fig. 7 Relationship diagrams between longitudinal force and maximum bending in beam and support rigidity

$l = 3$ m, $h = 0.2$ m, $b = 0.1$ m in size, assuming the following material characteristics: $E = 25000$ MPa and $R_e = 25$ MPa.

Figures 5 and 6 show the dependence between support stiffness and the relative load capacity of the beam (i.e., referring to the capacity of a beam where no longitudinal force emerges; at $k' = 0$). The graphs were made for different proportions of R_t / R_c. The calculations were performed according to the linear (Figs.5a, 6a) and non-linear (Figs. 5b, 6b) theories using the previously obtained equations. The strength of compression (R_c) or tension (R_t) the material attained in the extreme fibres of the beam, in the most strained sections, has been accepted as the criterion of capacity exhaustion. At the same time it was assumed that $R_c = R_e$.

Figure 7 presents a graph of the dependence between the longitudinal compression force N emerging in the beam, the value of the bending moment in the middle section of beam, M_{max} and the stiffness of the supports k' (at a fixed load on the beam: $q = 20$ kN/m or $P = 30$ kN).

On the other hand, Figure 8 illustrates the differences in the distribution of the bending moments in the beam, resulting from the application of different theories, assuming an infinite stiffness of the supports ($k' = \infty$). Similar differences, yet referring to deflections, are presented by graphs in Fig. 9.

Figure 10 shows the effect of the horizontal constraint stiffness k' on value of the beam deflection rise.

Conclusions

As it may be inferred from the graphs (Figs. 5 and 6) a horizontal support of the beam's ends produced obvious advantages. Working under the conditions causing a longitudinal compression force the beam shows a higher load-bearing capacity and lower deflections (Figs. 9, 10) than a beam working under "normal" conditions. Here, the key issue is the stiffness of the system of horizontal constraints k'. The appearance of the longitudinal compression force results from these constraints. When the support stiffness is close to zero, we obtain the solution (capacity) convergent with the solution relating to a merely bent beam. When the support stiffness increases, the load capacity of the beam grows as well.

Analysing the graphs in Fig. 5 (for a uniformly loaded beam) and Fig. 6 (for a beam loaded with a concentrated force), we see that, regardless of the R_t / R_c ratio, there initially exists a range in which, as stiffness k' grows, is an intense (and almost linear) increase in capacity. Then, not with standing a further increase in support stiffness, the capacity increases much more slowly, asymptotically tending toward a certain value. Hence, after having reached a certain support stiffness value its further increase is ineffective. Therefore, a very high stiffness of the horizontal support constraints is not necessary in order to obtain a significant reinforcing effect. It suffices for these constraints to have a certain small (and finite) stiffness and the benefit issuing from the capacity increase is already considerable. Especially for the beam in question, uniformly loaded and made of a material with a $R_t / R_c = 0.1$ ratio, at stiffness k' approximately equal to half of its compression strength ($k' = \frac{E \cdot A}{2 \cdot l}$), the capacity is already about twice as high, whereas in the case of infinitely stiff supports ($k' = \infty$) it is about 2.6 times as high as at bending without the longitudinal force effect (Fig. 5).

The capacity due to horizontal constraints is particularly visible when the material the beam is made of has a low tensile strength (e. g., concrete!). In this case, a high relative growth of beam capacity stems from the fact that the longitudinal force mark-

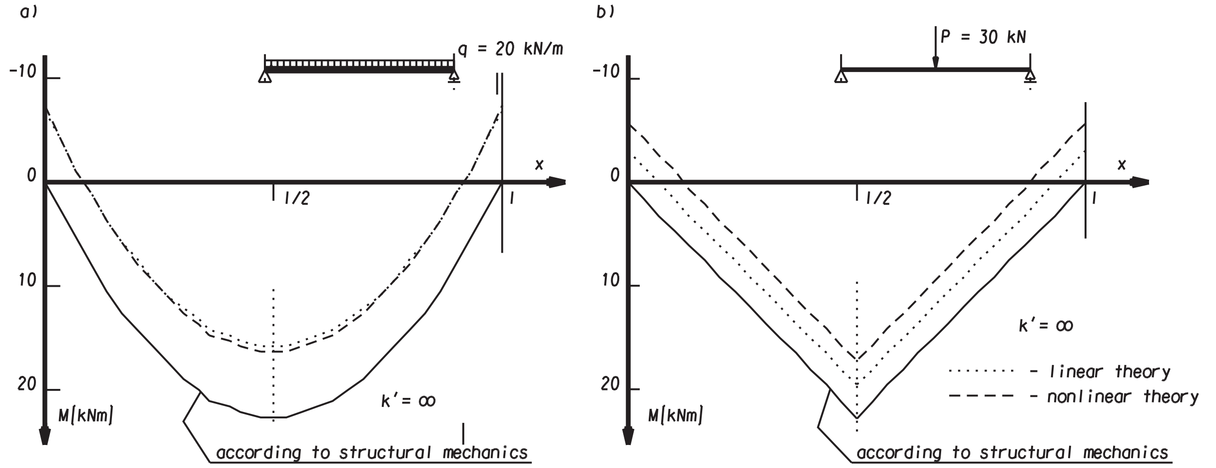

Fig. 8 Changes of bending moment distribution caused by horizontal support constraints

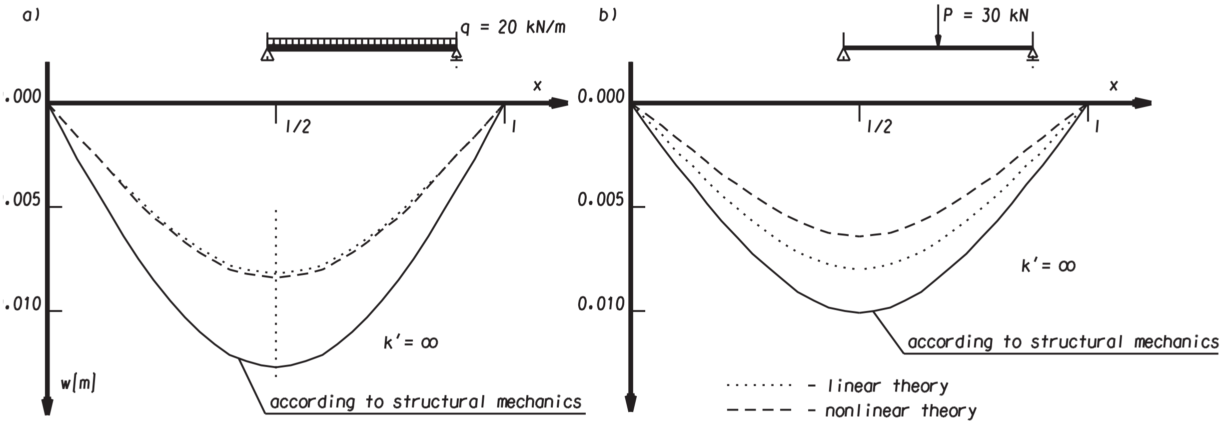

Fig. 9 Changes of beam deflection line caused by horizontal support constraints with infinite stiffness

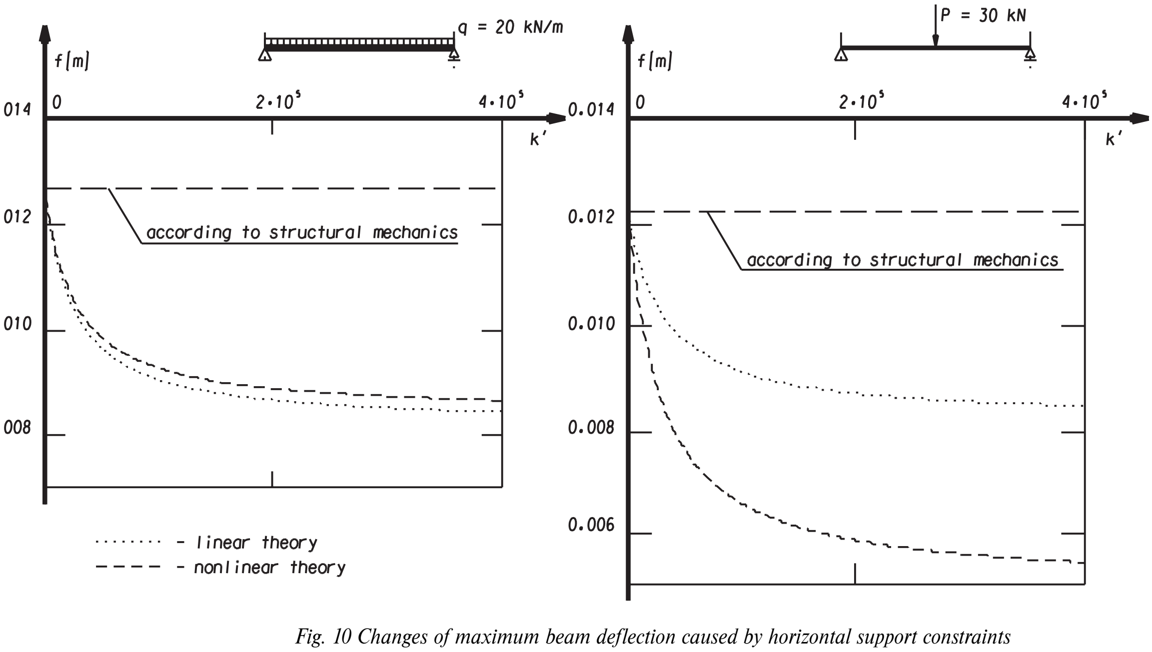

Fig. 10 Changes of maximum beam deflection caused by horizontal support constraints

edly reduces tensile stresses which, when the tensile strength of the material is low, are decisive in attaining the critical capacity of the beam (according to the capacity exhaustion criterion previously defined). On the other hand, if we deal with material whose tensile strength is equal or close to its compression strength, the capacity exhaustion occurs when the stresses on the compressed part of the section attain the value of compression strength. Despite being a compression force, the longitudinal force also decreases the compression stresses, though to a much smaller extent. This happens because force N, acting on eccentricity e_0 in relation to the gravity centre of the section, reduces the bending moments in the beam span (Fig. 8). The decrease of stress due to the reduction of bending moments is bigger than the stresses caused by compression. In consequence, the resultant compression stresses (caused by force N and the bending moment M) in the particular sections are lower than in the beam in which the longitudinal force does not occur. This allows the beam, whose ends are supported by horizontal frames, to transfer a higher load. When $R_t \approx R_c$, the capacity increase is generally not so high; it does not exceed 25 percent at uniform loads and 20 percent at concentrated loads (for $k' = \infty$).

In addition, it is worth noticing that in reality the reserves are even bigger in a beam supported at its ends with horizontal constraints and made of a low tensile strength material. We assumed, in the presented examples, the moment of capacity exhaustion to be equal to the stresses attaining compression or tensile strength in the extreme fibres of the most strained section; tensile stresses are decisive in $R_t << R_c$. Such a criterion is justified for a beam supported in a way not causing the emergence of the longitudinal force in it (for $k' = 0$, Fig. 1a) and when the material, if stretched, shows brittle properties - this is when a crack in the span is tantamount to damage. On to the contrary, if the beam is supported as shown in Fig. 1b, a crack in the span, due to the stresses exceeding tensile strength, will not damage the element. Following the cracking (in the most strained section) a quasi-joint will form and the structure will transform itself into a three-joint system. Its geometric invariability will be ensured by the horizontal support constraints. For this reason, the beam supported in such a way may efficiently transfer loads acting on it. Disregarding the capacity of the supports, and thus their ability to transfer the strutting forces acting on them, the destruction of such a system may occur in two-fold: by compression strength exhaustion (plasticization) of the material in the middle joint or due to the structure transforming itself into a mechanism. The transformation of a three-joint system into a kinematics chain will occur at big vertical displacements of the middle joint. This might be caused either by a low stiffness of the horizontal support constraints, k', or low material stiffness coefficient E. Under such circumstances, the reserve of an extraelastic load capacity is practically insignificant. Otherwise, of the two above mentioned conditions occur much later than the exhaustion of the material's tensile strength in the phase of the elastic work of the beam. Nonetheless, the equations for the two load regimes, presented in this paper, do not allow us to estimate the real (limiting) capacity of the beam as they were derived in accordance with the theory of elasticity based on the assumption of material continuum.

There will be a negligible inaccuracy if the capacity of a uniformly loaded beam is calculated according to the 1st order theory. The difference between the capacity calculated according to the linear and non-linear theories is about 3 percent. The difference is significant however, in the case of a beam loaded with a concentrated force. It becomes bigger with a decrease in the R_t / R_c ratio. The consequence of applying the linear theory leads to a remarkable underestimation of the beam capacity (in certain cases as big as 40 percent).

The way of supporting the beam for the purpose of hindering the rotation of the support section also leads to a considerable limitation of beam deflections (Figs. 9, 10). Initially as the support stiffness grows, they are remarkably reduced, even by more than 30 %. A further increase of this stiffness becomes, as in the case of load capacity, ineffective (Fig. 10).

Summary

To reiterate: the condition necessary to produce a longitudinal force in an element being bent comprises stiff horizontal supports capable of counteracting the pressure of beam ends or slab edges. Such conditions occur fairly often in real engineering structures, chiefly in those made of reinforced concrete. Primarily, they occur in uni-or bidirectionally bent slabs of slab-rib systems (Fig. 11), and also other floor systems such as slab or column-slab structures. In such systems, the role of the horizontal constraints counteracting outward slab expansion is assumed by the ribs surrounding the slab, spandrel beams and the neighbouring panels of the floor slab. The phenomenon of reinforced concrete slabs displaying compression forces is known in literature as the compressing membrane effect [1], [2] and has been investigated in world research centres. The beam with its ends protected from displacements can be exemplified by a lintel beam supported on the wall.

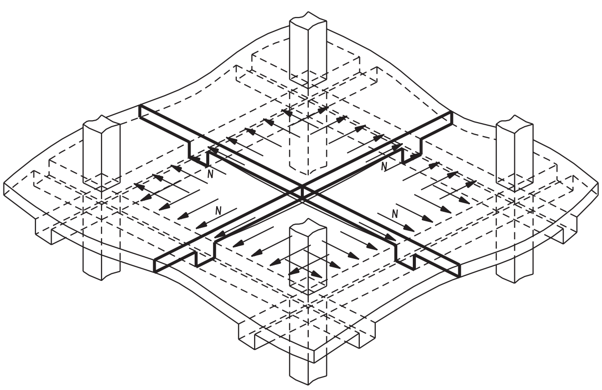

Fig. 11 Example of support strutting by plate in real structure

It is obvious that the equations here derived cannot be applied to estimate reinforced concrete beams (or slabs). This is because they omit to include the rheological phenomena in concrete, the concrete-specific $\sigma - \varepsilon$ dependence or other composite-specific features of reinforced concrete (e. g., cracking). Contraction and creeping undoubtedly reduce the strengthening effect. The increase in the load capacity of the elastic beam (Figs. 5 and 6) caused by the horizontal support of its ends, encourages us to continue theoretical and practical studies on reinforced concrete beams and slabs. Especially inciting to use this effect in concrete structures is the fact that such materials as concrete (i.e., of low tensile strength) offer the most noticeable advantages.

A detailed investigation of this problem will make it possible to precisely calculate the load capacity of these elements and, by the same token, exploit the inherent reserves. It will open the way to a better determination of the (especially horizontal) effects of a slab or beam on other structural elements surrounding it and the consequences of the effects upon these elements, in the form of additional internal forces.

Notation

Symbols used in this paper:

b – width of beam cross-section;
c_0 – range of compressed zone in the support section (Fig. 2);
$e_0 = \frac{1}{2}(h - c_0)$ – longitudinal force eccentricity;
E – elasticity modulus;
h – height of the beam cross-section;
J_y – section inertia moment in relation to y-axis (Fig. 2);
k_0 – stiffness characteristic of support;
M_0 – support moment;
$M_q(x)$ – bending moment induced by transverse load q;
$M_N(x)$ – bending moment induced by longitudinal force N applied on eccentricity e_0 in relation to section gravity centre;
N – longitudinal force;
P – force concentrated at the span centre,
R_c – compressive strength;
R_e – yield stress;
R_t – tensile strength;
t_0 – horizontal displacement of the support;
t_{s0} – horizontal displacement of beam support section gravity centre due to longitudinal deformations of the beam;
Σt – sum of support horizontal displacements;
$w(x)$ – beam deflection;
z – distance between beam fibre and section gravity centre;
ε_x – longitudinal unit strain in the beam;
φ_0 – rotation angle of the support section of the beam;

$$\beta = \sqrt{\frac{N}{EJ_y}} .$$

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[2] PARK, R.: *"The Ultimate Strength and Long Term Behaviour of Uniformly Loaded Two- Way Concrete Slabs with Partial Lateral Restraint at all Edges"*, Magazine of Concrete Research (London), V.16, No. 48, Sept. 1964, pp. 139-152.