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# VÝPOČTOVÝ MODEL A ANALÝZA HRANIČNÝCH PRVKOV PRE ŽELEZOBETÓNOVÉ PANELY S TRHLINAMI

## MODELING AND BEM ANALYSIS OF REINFORCED CONCRETE CRACKED PANELS

*Príspevok uvádza postup stanovenia rozdelenia napätí v železobetónovom paneli s trhlinami.Väzko-elastický model železobetónovej dosky sa odvodil Gurtinovou variačnou metódou v priestore všeobecnej funkcie. Výsledné diferenciálne rovnice sú prezentované vo forme vektorových všeobecných funkcií. Diskontinuita všeobecného deformačného vektora, preklz po strate súdržnosti ako aj dotvarovanie betónu sú zohľadnené v popísanom výpočtovom modeli. Uvedené sú tiež numerické výsledky aproximatívneho riešenia použitím metódy hraničných prvkov.*

*In this paper the method of distribution solution of RC (reinforced concrete) cracked of the plane stress plates (panels) is shown. The viscoelastic RC cracked plate model has been derived by the variation method of Gurtin in the class of the general function. The total differential equations in the two-dimensional general vector functions are shown. In this model the effect of the discontinuity general deformation vector, bond slip and concrete creep are taken into account. The numerical results of approximate solutions with the method of boundary elements (BEM) are shown.*

### 1. Introduction

The RC concrete plates are non-homogeneous. Therefore, the response of such heterogeneous structures and additional defects caused by cracks in concrete to applied actions is generally non-linear, due to nonlinear constitutive relationships of the materials, known as mechanical nonlinearity and to second order effects of normal forces, known as geometrical nonlinearity. Defects in form of cracks treated as continuous functions, which are usually based on the continuum mechanics approach, give unsatisfied solution because of summation of assumption errors and solution errors. Therefore, the proper mathematical modeling of panels is so important since the final error appears solely in a solution phase.

This paper contains a mathematical model of reinforced concrete panels formulated in terms of general functions. The physical hypothesis about the discontinuous change of the displacement vector, caused by the cracking of the extension zone in the concrete, with the assumption of rotating cracks, is included in the model. The assumptions of the distribution theory of Schwartz [1] afford possibilities for precise mathematical description of the crack discontinuity of the panels.

### 2. Differential equation for displacement

The arbitrary plane stress plate is considered. The plate has arbitrary homogeneous boundary conditions and is arbitrarily forced. The region of plate $\Omega$ is divided by the curve $\Lambda$ which means the crack, in two zones $\Omega_1$ and $\Omega_2$ with bound $\partial\Omega_1$ and $\partial\Omega_2$. The curve $\Lambda$ has two ends $\Lambda_1$ and $\Lambda_2$. The normal external direction cosines of the edge $\Lambda$ of regions $\Omega_1$ and $\Omega_2$ have different signs. The considered model can be easily generalized to any number of cracks $\Lambda$. [M1]

The discontinuous variation problem of the surface integral for the displacement of the viscoelastic plane stress plate $\Omega$ was solved.

The equilibrium equations in the form: $div\,\boldsymbol{S} + \boldsymbol{b} = \boldsymbol{0}$ (where $\mathbf{b}$ – body forces), and geometrical relations in the form:

$E = \frac{1}{2}(\nabla u + \nabla u^T) = \tilde{\nabla} u$ are taken.

The set of the field equations is fulfilled in the space $\Omega x < 0, \infty)$, where $< 0, \infty)$ is the time interval. The initial condition of the strain' tensor has to be added $\boldsymbol{E}(\,.\,, 0) = \boldsymbol{E}^0$, for $t = 0$.

The physical law is taken as well-known Boltzmann's rule.

The constitutive law of defect is assumed to be represented as follows:

$$[\boldsymbol{u}(\boldsymbol{x})]_{\Lambda_1\Lambda_2} = \boldsymbol{r}(\boldsymbol{x}), \text{ with conditions}$$
$$\frac{\partial r}{\partial s}(\Lambda_1) = \frac{\partial r}{\partial s}(\Lambda_2) = 0, \qquad (1)$$

where $[\,.\,]\Lambda$ means the difference of the left and right side limit of the expression in square braces on the curve $\Lambda$.

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Here $\boldsymbol{r}(\boldsymbol{x})$ describes the density of the defect as a continuous function for $x \in \Lambda_1\Lambda_2$ and $[\boldsymbol{u}]_\Lambda = 0$ for $x \notin \Lambda_1\Lambda_2$. To satisfy the first condition of Eq. (1), the modeling on the crack edges by the dipole normal forces for the plate is assumed.

Equation (1) satisfies the compatibility condition in the crack, where the displacement vector has a jump on a bound of crack. Here the assumption of internal crack $\Lambda_1\Lambda_2$ was taken, which can be easily demonstrated. Hence, on the remaining part of the curve $\Lambda$ the condition $[\boldsymbol{u}]_\Lambda = 0$ is valid, for $x \notin \Lambda_1\Lambda_2$. Moreover, the second condition (1), in the essential way, completes the definition of the defect. Note that the constitutive law of the crack opening is expanded as a rule additionally valid in time.

The assumption of the jump of displacement vector $\boldsymbol{u}(\boldsymbol{x})$ was demonstrated in an experimental study, see [2]. The zone of plate $\Omega_1$ is connected with another one $\Omega_2$ by means of reinforcement bars appearing in the cracks. Hence, the edges of the cracks are not free from tensions at the points of connections. Outside the reinforcement points, on the remaining edge segments of the cracks, the boundary conditions should be equal to the conditions corresponding to the free edges. The density of defect, also known as the constitutive law of crack opening, is the function of tension vector $\boldsymbol{N}$ acting in the crack:

$$\boldsymbol{r}(\boldsymbol{x}) = \boldsymbol{r}^0(\boldsymbol{x}) - \boldsymbol{r}^1(\boldsymbol{x})\Im(\boldsymbol{N}(\boldsymbol{x}))\big|_\Lambda, \boldsymbol{x} \in \Lambda_1\Lambda_2 \qquad (2)$$

Here $\boldsymbol{r}^0$ describes residual general deformations, whereas $\boldsymbol{r}^1\Im(\boldsymbol{N})$ describes the elastic general deformations. The components $\boldsymbol{r}$ of Eq. (2) are given from RC element tests, from general assumptions of crack theory, and equilibrium conditions in the crack.

The discontinuous viscoelastic variation problem of the Gurtin type was solved. The equilibrium equations, Boltzmann constitutive law, strain equations and the initial condition are assumed to be represented by the well-known theory of elasticity relations. We are looking for the extreme of the functional of strain energy with a set of permissible displacement value $\boldsymbol{u}(\boldsymbol{x})$.

The searching function $\boldsymbol{u}(\boldsymbol{x})$ is in the class of the function $\boldsymbol{u} \in C^2(\Omega/\Lambda)$ (for $\boldsymbol{x} \in \Lambda$ function $\boldsymbol{u}(\boldsymbol{x})$ has singularity).

Applying Green's transformation with material relations, field relations and the constitutive law of cracks taken from Eq. (1), additionally using a functional description with $\delta$ of Dirac's type, the resulting general differential equation of viscoelastic RC cracked plate in plane stress, appropriate boundary, compatibility and initial conditions, respectively, can be written as follows:

$$[\psi 1 * \nabla^2 + \frac{1}{2}(\psi_1 + \psi_2) * grand\ div]\boldsymbol{u}(\boldsymbol{x},t) + 1 *$$
$$* \boldsymbol{b}(\boldsymbol{x},t) - div\,\boldsymbol{F} = -\widetilde{N}(\boldsymbol{r}(\boldsymbol{x},t)\delta_\Lambda) + [1 * \widetilde{\boldsymbol{p}}(\boldsymbol{x},t) -$$
$$- \widetilde{N}(\boldsymbol{u}(\boldsymbol{x},t))]\delta_{\partial\Omega_1} + \widetilde{N}[(\widetilde{\boldsymbol{u}}(\boldsymbol{x},t) - \boldsymbol{u}(\boldsymbol{x},t))]\delta_{\partial\Omega_2}, \qquad (3)$$

where:

$$\boldsymbol{F} = \psi_1 * \boldsymbol{E}^0 + \frac{1}{2}(\psi_2 - \psi_1) * \mathbf{1}\, tr\, \boldsymbol{E}^0. \qquad (4)$$

The viscoelastic operator $\widetilde{N}$ corresponds to the analogy of surface tension from the theory of elasticity as follows:

$$\widetilde{N}(.) = -[\boldsymbol{F} - (\psi_1 \widetilde{\nabla} + \frac{1}{2}(\psi_2 - \psi_1) * div)(.)]\boldsymbol{n}. \qquad (5)$$

Here the functions $\psi_1$ and $\psi_2$ are the functions of relaxation, $\boldsymbol{n}$ represents the normal vector external to the edge $\partial\Omega$ and $f(t) * * g(t)$ means the convolution rule.

Note that the final solution (3) is similar to the elastic solution achieved by Minch, see [3], where the difference occurs only for the relaxation function, with $\lambda$ and $\mu$ as a time dependent function.

The solution of Eq. (3) is possible with the help of the elastic solution as a first approximation of the viscoelastic solution. It denotes the solution of the "associated" elastic problem $\boldsymbol{u}(\boldsymbol{x},t)$ from the elastic solution. This method can be used for solving the convolutions' of the viscoelastic static problem of the RC cracked plane stress as follows:

$$\boldsymbol{u}(\boldsymbol{x},t) = \int_0^t \partial\frac{\boldsymbol{u}(\boldsymbol{x},\tau)}{\partial\tau}\varphi(t-\tau)d\tau, \qquad (6)$$

where $\varphi$ is the function with the combination of relaxation and creep functions.

## 3. Modeling by Boundary Element Method

Deformation behaviour depends on the history of the loading as well as the non-linearity of the material properties. Hence, the equations and definitions of the boundary element method in the rate form were assumed. According to the small strain theory, the total strain rate for an inelastic problem can be divided into an elastic and inelastic part of the total strain rate tensor. Herein, the inelastic strain means any kinds of strain field that can be considered as initial strain, i.e. plastic or viscoplastic strain rate, creep strain rate, thermal strain rate and strain rate due to other causes. So, now we can write the equations of the considered problem in terms of non-linear BEM formulations for fictitious traction vector $\boldsymbol{p}$ and body forces' $\boldsymbol{b}$, finally leading to the initial stresses $\sigma^p$:

$$\boldsymbol{H}\dot{\boldsymbol{u}} = -\boldsymbol{A}\dot{\boldsymbol{p}} + \boldsymbol{B}\dot{\sigma}^p + \boldsymbol{F}\dot{\boldsymbol{x}} + \boldsymbol{Q}(\dot{\boldsymbol{x}}), \qquad (7)$$

where $\boldsymbol{u}$ means displacement vector, $\boldsymbol{x}$ is the vector of unknown edge traction, $\boldsymbol{p}$ means the vector of fictitious traction, while $\sigma^{\mathbf{p}}$ is the vector of initial stresses. Here the matrices $\boldsymbol{H}$ and $\boldsymbol{A}$ are the same as for elastic analysis, matrix $\boldsymbol{B}$ due to the inelastic stress integral, matrix $\boldsymbol{F}$ refers to the fundamental function caused by the forcing traction with vector $\boldsymbol{x}$, i.e. modeling the density of the crack opening, while the matrix of bond $\boldsymbol{Q}$ includes creep, bond-slip relations and other displacements due to the aggregate interlock and dowel action of reinforcement in the crack, related to displacement $\boldsymbol{u}$.

## 4. Incremental computations

Equation (7) must be solved numerically with iterative and incremental techniques. Iteration results are due to the fact that the right side of Eq. (7) depends directly on functions ***u***. In addition, function ***u*** depends indirectly on the physical law i.e. Eq. (2). The incremental computation is caused by the rate form of Eq. (7). The modified Newton-Raphson method was applied to the iteration and incremental computations.

The concrete properties should be included in the biaxial domain. The biaxial tests of Kupfer, see [4], for short time loading and proportionally increasing load, proved to be the most reliable. Link, see [5], developed an incremental formulation for the tangent stiffness of the concrete on the basis of Kupfer's tests. The concrete physical law of Link, see [5], was used in the computation of the planar structure within the presented method.

The creep of concrete was taken from the Bažant's and Panula, see [6], rheological model with the creep function as below ($f_c'$ means the 28 days compressive strength of concrete):

$$J(t,\tau) = \frac{1}{E_o(f_c')}\,[1 + \varphi_1\,(f_c')(\tau^{-m(f_c')} + \alpha(t-\tau)^{n(f_c')}]. \quad (8)$$

The $\sigma - \varepsilon$ relation of steel bars was taken as elasto-plastic relation from uniaxial tests.

After the cracking of the concrete, the tensile forces in the cracked area are transmitted by bond to the reinforcement that consists of steel bars. Along the segments of broken adhesion the steel bar interacts with the concrete through the tangential stresses distributed on the perimeter of the bar. The slip $\Delta$ is defined as a relative displacement between the reinforcement bars and surrounding concrete. The increment of tensile stresses in the steel bar was approximated by the third-degree curve. Hence, the tangential stresses $\tau$ and bond-slip relationships, as representation of the stiffness of the bond, have been found to be in agreement with the tests of Dörr and Mehlhorn, see [7], i.e. the second-degree distribution along the segment $l_f$, where $l_f$ means distance between cracks.

The programme of the BEM Analysis, named *PLATE*, for two-dimensional problems was designed. The *PLATE* analysis includes the procedures of: *Modeling System (MS)*, where the model of construction is built, the *Analysis Module (AM)* where the problem with iterative and incremental method is solved and finally using *MS* the results are obtained *(RES)*. The iterative and incremental techniques used in the *(AM)* connect all material properties with crack physical laws and edges conditions.

The time-dependence of bond in the loaded state exhibits a similar behaviour as concrete in compression, see[8]. The presupposition similar to the linear creep theory of concrete in compression is used for bond creep with bond creep coefficient $\varphi_b$. Naturally in accordance with $\tau - \Delta$ relationship of Dörr and Mehlhorn, see [7], bond creep cannot be described by linear theory. The model of Rotasy and Keep, see [8], was applied to describe the creep of the bond in the cracked concrete.

The development of "rotating cracks" is considered as single cracks treated as the boundary element where the direction of the crack has to be assumed in accordance with the previous step of the main direction of the tensile stresses.

## 5. Numerical example

The results of a simply supported square panel *WT3* (1.6 × × 1.6 m), tested by Leonhardt and Walther, see [9], were taken to check BEM base solution for plane stresses (the comparison of the test and calculation results is included only in Fig. 1). The panel was reinforced horizontally in a different way for the bottom and top part. The bottom zone ($f_{y1}$) had 2$\phi$8 mm bars each 6 cm fixed in 4 rows, the top zone and vertical bars ($f_{y2}$) were 2$\phi$5 mm each 26 cm.

Fig. 1 shows the comparison of the calculated load-midspan deflection relations with the test results of panel *WT3*. Figure 2 depicts the propagation of cracks and numerical calculated width of cracks under different loading steps and time. Fig. 3 presents the midspan crack width of *WT3* panel along the dimensionless height of the panel while Fig. 4 demonstrates the dependence of the time and loading levels on the crack width $a_f$.

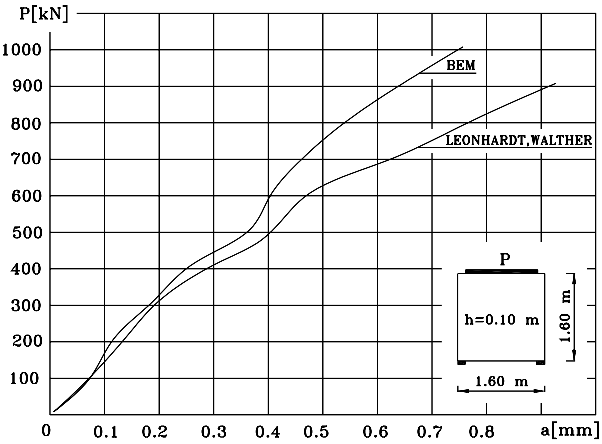


*Fig. 1. Comparison of calculated load-midspan deflection relations with the test result of panel WT3 [9]*

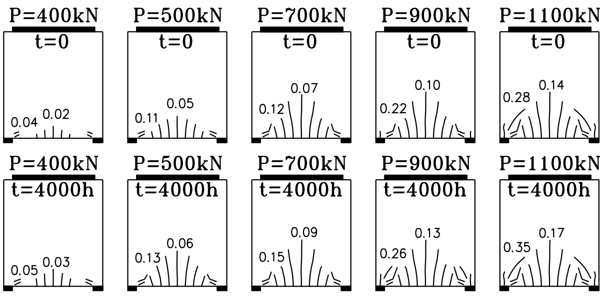


*Fig. 2. The propagation of cracks and width of cracks under different loading levels and time*

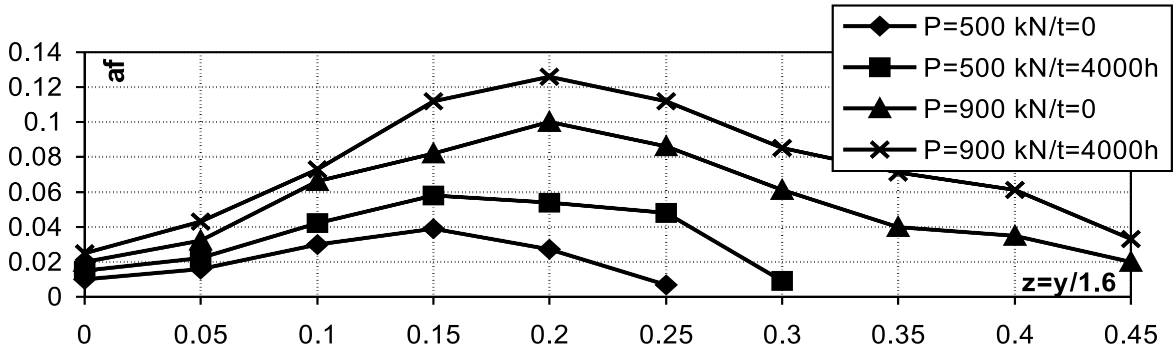


*Fig. 3. The midspan crack width af of WT3 panel along the dimensionless height of the panel*

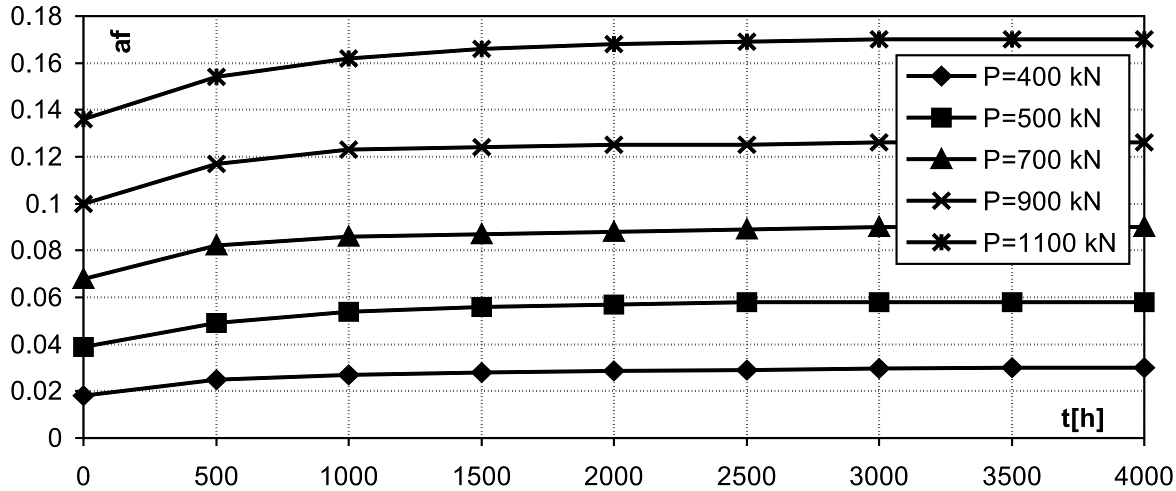


*Fig. 4. The dependence of the time and loading levels on the midspan crack width af*

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