https://doi.org/10.26552/com.C.2003.4.66-70

KOMUNIKÁCIE
COMMUNICATIONS

Juraj Smieško *

EXPONENTIAL MODEL OF TOKEN BUCKET SYSTEM

In the presented paper we use the Theory of Markovov Chains for simulation of a simple model of the Token Bucket System (TBS). In the ***first section*** *we deal with a steady-state analysis of the TBS with Exponential model of On-Off Source of human speech. We develop recurrent formulas for state probabilities of the TBS. In the* ***second section*** *we use real parameters of VoIP and we calculate characteristic of the TBS. In the* ***last section*** *the values of probability of a loosing packet are approximated by more types of regression functions. It is shown that the cubic approximation is the most efficient. Using this function we can directly compute values of probability of a loosing packet.*

1. Steady-state Analysis

The Token Bucket System is related to VoIP problems. We have formed the steady-state analysis of VoIP under the Token Bucket Control. Our main problems will be to compute the probability characteristics of TBS and to find relation between probability of a loosing packet and the bucket depth. At first we will analyze the work of TBS in general (Fig. 1).

ON: $T_1 \sim f_1(t) = \alpha e^{-\alpha t}$

$$ET_1 = \frac{1}{\alpha} = 227\ ms \quad \alpha = 0.00441 ms^{-1}$$

OFF: $T_2 \sim f_2(t) = \mu e^{-\mu t}$

$$ET_1 = \frac{1}{\mu} = 596\ ms \quad \alpha = 0.00168 ms^{-1}$$

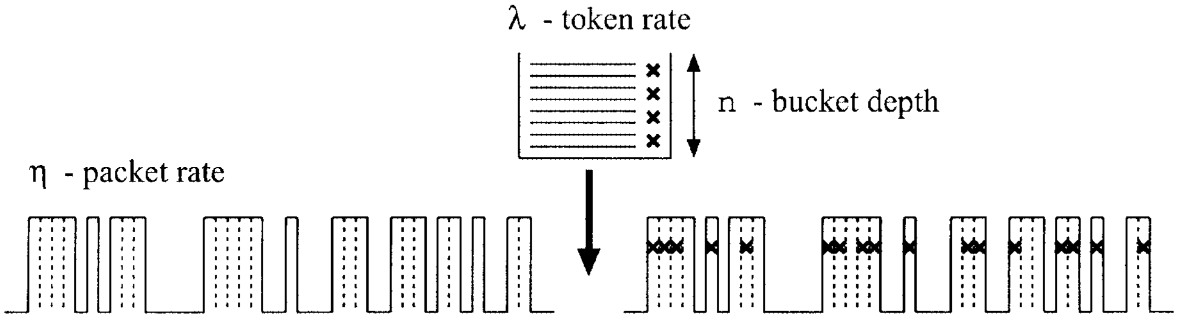


Fig. 1

There is a flow of packets entering the TBS. The Token Bucket System is generating tokens (marks) and then it marks each packet with one of them. Only marked packets will go through the network. For practical reasons we can assume a limited bucket with depth "*n*". In case the bucket is empty (there are no tokens), TBS cannot mark any packet, which is lost then. Let P_{lst} be probabilities of this random phenomenon and we will call it "probability of loosing packet".

We will deal with simple Exponential model of On-Off Source of human speech. This Source has two states. The state On is period of "speech" and the Off state is period of "silence". Between these states, the source switches randomly. Talk-spurt duration is modelled by variant T_1 and pause duration is modelled by variant T_2. The values of distribution parameters were gained from [1].

When the Source is in On state it starts generating a flow of packets, which represents human speech. The packet rate in usual VoIP systems (using G.729A) is 50 p/s. We will assume that the flow of packets is modelled by Poisson process $N_P(t)$ with rate $\eta = 0.05\text{p/ms}$.

In general there can be any token rate. Usually it is the same as an average number of packets entering the TBS. Let P_{ON} and P_{OFF} be probabilities that the Source is in state On or Off. The flow of tokens will be modelled by Poisson process $N_T(t)$ with rate $\lambda = \eta \,.\, P_{ON} + 0 \,.\, P_{OFF}$.

We will construct a transition diagram of Markov model of On-Off Source (Fig. 2):

* **Juraj Smieško**
Department of InfoCom Networks, Faculty of Management Science and Informatics, University of Žilina

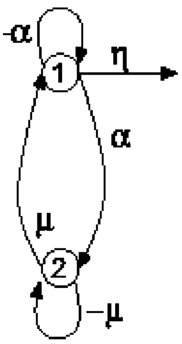


Fig. 2

Let Q_0 be a rate matrix for this chain. We compute state probabilities π of a steady-state chain by balance equations:

$$Q_0 = \begin{pmatrix} -\alpha & \alpha \\ \mu & -\mu \end{pmatrix} \quad \pi = (\pi_1, \pi_2),$$

$$\pi \cdot Q_0 = 0 \;\Rightarrow\; \pi = \left(\frac{\mu}{\alpha+\mu}, \frac{\alpha}{\alpha+\mu}\right)$$

We compute probabilities and token rate for real parameters of VoIP:

$\pi = (0.27586,\ 0.72414)$

$\mu = 0.05 p/ms$. $P_{ON} = 0.05 p/ms$. $\pi_1 = 0.01379\ p/ms$

In the Token Bucket System we have three elementary random phenomena, entering the packet, generating token and switching between On-Off states in Source. We have assumed that each of them is Poisson process, and because of that we can model this TBS by Markov chain.

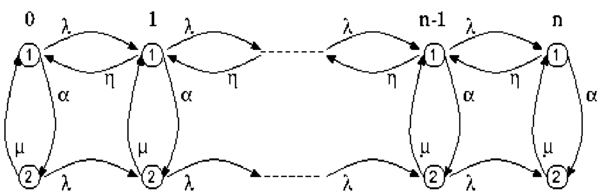


Fig. 3

Probability of an increasing number of tokens in the bucket about one during "short" time Δt is

$$P(N_T(\Delta t) = 1) = \lambda\Delta t \cdot e^{-\lambda\Delta t} = \lambda\Delta t + o(\Delta t)$$

If the bucket contains k tokens, probability of a decreasing number of tokens in the bucket about one during "short" time Δt is

$$P(N_P(\Delta t) = 1) = \eta\Delta t \cdot e^{-\eta\Delta t} = \eta\Delta t + o(\Delta t)$$

If there is an arriving packet and the bucket is empty, we cannot mark it and this packet is lost. Probability of loosing packet was named P_{lst}.

We will construct a transition diagram of the whole system (Fig. 3). Component columns of diagram refers to a number of tokens in the token bucket. We have assigned this as "k-level" for $k = 0, ..., n$. For easy reading loops in the transition diagram are omitted:

Let $X_k(Y_k)$ be a probability that in the TBS are k tokens and Source is in state On (Off). Let P_k be a probability of k-level. We see that $P_k = X_k + Y_k$ and $P_{lst} = X_0$. Taking from the Theory of Markov Chains [2] we can write the following equations for state probabilities of the steady-state chain:

$$\begin{aligned}
\text{for } k = 1 \quad & 0 = -(\lambda + \alpha)X_0 + \mu Y_0 + \eta X_1 \\
& 0 = \alpha X_0 - (\lambda + \alpha)Y_0 \\
& \vdots \\
\text{for } k = 2, ..., n-1 \quad & 0 = \lambda X_{k-1} - (\lambda + \alpha + \eta)X_k + \mu Y_k + \eta X_{k+1} \\
& 0 = \lambda Y_{k-1}\, \alpha X_k + \mu Y_k - (\lambda + \mu)Y_k \\
& \vdots \\
\text{for } k = n \quad & 0 = \lambda X_{n-1} - (\alpha + \eta)X_n + \mu Y_n \\
& 0 = \lambda X_{n-1} + \alpha X_n - \mu Y_n
\end{aligned}$$

We will solve the steady-state equations $Q'p = 0$, where $p = (Y_n, X_n, ..., Y_k, X_k, ...\ Y_0, X_0)$ and matrix Q' is formed as:

$$Q' = \begin{pmatrix}
-\mu & \alpha & \lambda & 0 & \ldots & 0 & 0 & 0 & 0 \\
\mu & -(\alpha+\eta) & 0 & \lambda & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & -(\lambda+\mu) & \alpha & \ldots & 0 & 0 & 0 & 0 \\
0 & \eta & \mu & -(\lambda+\alpha+\eta) & \ldots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & -(\lambda+\mu) & \alpha & \lambda & 0 \\
0 & 0 & 0 & 0 & \ldots & \mu & -(\lambda+\alpha+\eta) & 0 & \lambda \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & -(\lambda+\mu) & \alpha \\
0 & 0 & 0 & 0 & \ldots & 0 & \eta & \mu & -(\lambda+\alpha)
\end{pmatrix}$$

We modified the rate matrix Q' to the equivalent triangular matrix and we used a normalization condition for state probabilities $\sum_{k=0}^{n}(X_k + Y_k) = 1$

$$\begin{pmatrix}
-\mu & \alpha & \lambda & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & -\eta & \lambda & \lambda & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & -(\lambda+\mu) & \alpha & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\eta & \ldots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & -(\lambda+\mu) & \alpha & \lambda & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & -\eta & \lambda & \lambda \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & -(\lambda+\mu) & \alpha \\
1 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 1
\end{pmatrix} \cdot \begin{pmatrix} Y_n \\ X_n \\ Y_{n-1} \\ X_{n-1} \\ \vdots \\ Y_1 \\ X_1 \\ Y_0 \\ X_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

We gained the recurrent formulas for state probabilities:

$$Y_0 = \frac{\alpha}{\lambda+\mu} X_0 \,, \quad X_k = \frac{\lambda}{\eta}[Y_{k-1} + X_{k-1}] \quad \text{for } k = 1, ..., n$$

$$Y_k = \frac{\alpha}{\lambda+\mu} X_k + \frac{\lambda}{\lambda+\mu} Y_{k-1} \quad \text{for } k = 1, ..., n\,,$$

$$Y_n = \frac{\alpha}{\mu} X_n + \frac{\lambda}{\mu} Y_{n-1}$$

We can reduce formulas for Y_k and Y_n to:

$$X_k = \frac{\lambda}{\lambda+\mu}\left[\frac{\alpha+\eta}{\eta} Y_{k-1} + \frac{\alpha}{\eta} X_{k-1}\right] \quad \text{for } k = 1, ... n,$$

$$Y_n = \frac{\lambda}{\mu}\left[\frac{\alpha+\eta}{\eta} Y_{n-1} + \frac{\alpha}{\eta} X_{n-1}\right]$$

2. TBS with real parameters of VoIP

To deduce explicit formula for probability of loosing packet $P_{lst} = P_{lst}(n) = X_0$ is difficult and impracticle, but there is no problem to compute the values of P_{lst} for real parameters of our TBS with Exponential On-Off Source:

$\alpha = 0.00441\ m.s^{-1} \quad \mu = 0.00168\ m.s^{-1}$

$\lambda = 0.01379\ p/ms \quad \eta = 0.05 p/ms$

Than recurrent formulas for state probabilities are formed as:

$Y_0 = 0.28507 \cdot X_0 \quad X_k = 0.27580 \cdot [Y_{k-1} + X_{k-1}]$

for $k = 1, ..., n$,

$Y_k = 0.97002 \cdot Y_{k-1} + 0.07862 \cdot X_{k-1} \quad$ for $k = 1, ..., n$,

$Y_n = 8.93231 \cdot Y_{n-1} + 0.72397 \cdot X_{n-1}$

For example, the reader can see the difference between $n = 4$ and $n = 5$:

n	X_k	Y_k	P_k
0	0.14880	0.04242	0.19122
1	0.05274	0.05285	0.10559
2	0.02912	0.05541	0.08453
3	0.02331	0.05604	0.07935
4	0.02188	0.51742	0.53931

n	X_k	Y_k	P_k
0	0.13803	0.03935	0.17738
1	0.04892	0.04902	0.09794
2	0.02701	0.05140	0.07841
3	0.02136	0.05198	0.07361
4	0.02030	0.05212	0.07242
5	0.01997	0.48027	0.50024

For real use its enough to have the bucket depth $n = 1, ..., 10$. Now we will increase the bucket depth n and calculate characteristics of models:

$P_{lst}(n)$ – probability of loosing packet or probability of empty bucket in time of arriving packet

$\lambda \,.\, P_{lst}(n)$ – average number of lost packets

P_n – probability of full bucket (bucket will refuse tokens)

EK – average bucket depth $\aleph = \dfrac{EK}{n}$ – token bucket usage

n	$P_{lst}(n)$	$\lambda \cdot P_{lst}(n)$	P_n	EK	$\aleph$
1	0.20367	2.809 p/s	0.73826	0.73826	73.8%
2	0.17796	2.454 p/s	0.64503	1.41634	70.8%
3	0.16163	2.229 p/s	0.58580	2.05572	68.5%
4	0.14880	2.052 p/s	0.53931	2.66994	66.7%
5	0.13803	1.903 p/s	0.50024	3.26649	65.3%
6	0.12874	1.775 p/s	0.46658	3.84967	64.2%
7	0.12064	1.664 p/s	0.43719	4.42225	63.2%
8	0.11349	1.565 p/s	0.41129	4.98619	62.3%
9	0.10715	1.478 p/s	0.38828	5.54295	61.6%
10	0.10148	1.399 p/s	0.36771	6.09369	60.9%

3. Relation between Probability of loosing packet and the Token Bucket depth

The most interesting characteristic is probability of loosing packet $P_{lst}(n)$. Its values will be approximated by regression functions $Y_I(n)$:

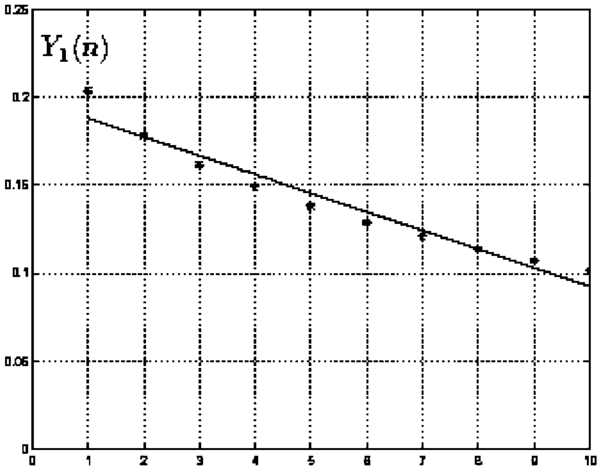
$Y_1(n) = an + b \qquad Y_2(n) = an^2 + bn + c$

$Y_3(n) = an^3 + bn^2 + cn + d \qquad Y_e(n) = ae^{bn}$

by the least squares method. We will measure the quality of approximation by square of sum residuals:

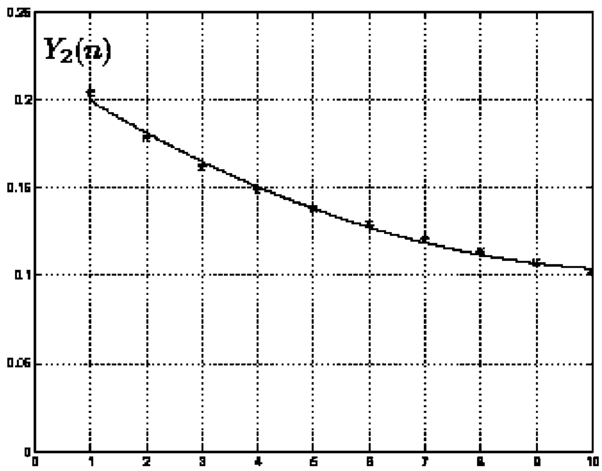
$$\mathfrak{E}_I = \sqrt{\sum_{n=0}^{10} [P_{lst}(n) - Y_I(n)]^2} \qquad I = 1, 2, 3, e$$

Approximation by the linear function: $Y_1(n) = -0.01060 \,.\, n + 0.19849$ with $\mathfrak{E}_1 = 0.02307$



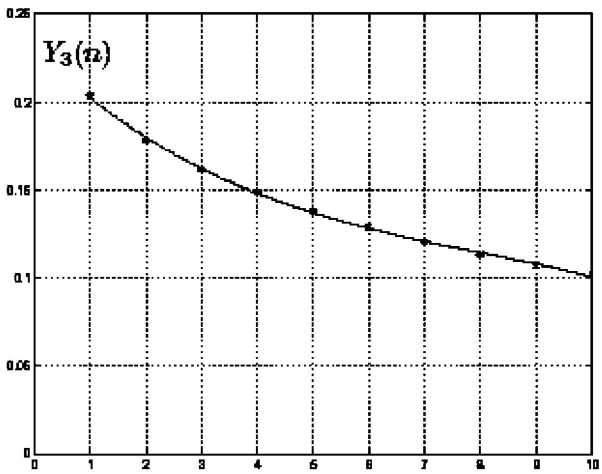
n	$Y_1(n)$	$P_{lst}(n)$ - $Y_1(n)$
1	0.18788	0.01579
2	0.17728	0.00068
3	0.16667	0.00505
4	0.15607	0.00726
5	0.14546	0.00743
6	0.13486	0.00611
7	0.12425	0.00361
8	0.11365	0.00015
9	0.10304	0.00411
10	0.09244	0.00904

Quadratic regression: $Y_2(n) = 0.00095 \,.\, n^2 - 0.02105 \,.\, n + 0.21937$ with $\mathfrak{E}_2 = 0.00753$



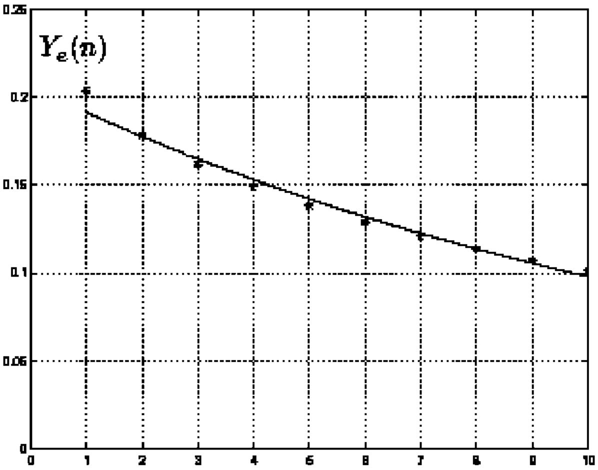
n	$Y_2(n)$	$P_{lst}(n)$ - $Y_2(n)$
1	0.19927	0.00440
2	0.18108	-0.00311
3	0.16477	-0.00315
4	0.15037	-0.00157
5	0.13787	0.00016
6	0.12726	0.00148
7	0.11856	0.00208
8	0.11175	0.00175
9	0.10684	0.00031
10	0.10383	-0.00235

Cubic regression: $Y_3(n) = -0.00012 \,.\, n^3 + 0.00297 \,.\, n^2 - 0.03039 \,.\, n + 0.22990$ with $\mathfrak{E}_3 = 0.00318$



n	$Y_3(n)$	$P_{lst}(n)$ - $Y_3(n)$
1	0.20237	0.00131
2	0.18004	-0.00208
3	0.16220	-0.00057
4	0.14809	0.00071
5	0.13697	0.00104
6.	0.12815	0.00060
7.	0.12084	-0.00020
8	0.11432.	-0.00083
9	0.10787	-0.00072
10	0.10073	0.00074

Approximation by the exponential function: $Y_e(n) = 0.20615 \, . \, e^{-0.07447}$ with $\mathfrak{E}_e = 0.01501$



n	$Y_e(n)$	$P_{lst}(n)$ - $Y_e(n)$
1	0.19135	0.01232
2	0.17762	0.00034
3	0.16487	-0.00325
4	0.15304	-0.00424
5	0.14206	-0.00403
6	0.13186	-0.00312
7	0.12240	-0.00176
8	0.11361	0.00012
9	0.10546	0.00169
10	0.09789	0.00359

We can see the cubic regression function is a very "good" approximation with maximum error $2{,}1 \cdot 10^{-3}$. If we are satisfied with this precision, we can exchange $P_{lst}(n)$ for $Y_3(n)$:

$$P_{lst}(n) \doteq -0.00012 \cdot n^3 + 0.00297 \cdot n^2 - 0.03039 \cdot n + 0.22990$$

Because of its practical use the exponential approximation is "better" (with maximum error $1{,}2 \cdot 10^{-2}$).

References

[1] ITV-T Recommendation P.59: *Articial Conversational Speech,* 1993
[2] PEŠKO & SMIEŠKO*: Stochastic Models of Operation Research (Stochastické Modely Operačnej Analýzy)*, Žilinská univerzita, 1999.