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ON ONE CLASS OF THE INFINITE NON-ABELIAN GROUPS

We present some properties of one class of the infinite non-abelian groups. We deal with a generalization of the INH and KI groups. Our main results are Theorem 1, 2, 3 and Theorem 4.

Key Words and Phrases: a group, a commutator of group, a locally graded group, p-quasicyclic group, a direct and semidirect product of groups, an extension of the group.

The description of groups defined by the systems of their subgroups was first described in the papers by Chernikov, S. N. and Kurosh, A. G (*RN* – groups, [5]) Chernikov dealt with an extension of the direct product of the finite number of the quasicyclic groups by the finite abelian group (ч – groups,[1]), and with the infinite non-abelian groups whose arbitrary infinite subgroup is the normal subgroup of the whole group (*INH* – groups, [1]). Subbotin, I. J. studied the groups *G* whose every subgroup of the commutator G' is a normal subgroup of *G* (*KI* – groups, [6]). In this paper we describe the infinite non-abelian groups with the finite nodal subgroup.

We use the standard designations of terminology of the theory groups. For example: $M \times N$ the direct product of the groups *M* and *N*, $\sum_{\in} X_i$, the direct sum of the groups X_i for all $i \in I$, $M \lambda N$ the semidirect product of the groups *M* and *N*, $M.N = \{mn; m \in M, n \in N\}$ the product of the groups *M* and *N*, G/A the factor group of *G* by *A*, $|G : N|$ the index of the subgroup *N* in a group *G*, $< a >$ the cyclic group generated by one element *a*, $< a, b, c >$ the group generated by the elements *a, b, c*, $H \lhd G$ *H* is normal in *G* or *H* is a normal subgroup of *G*, $G' = [G,G]$ the commutator of the group *G*, $Z(p^{\infty}) = \{x; x^{p^n} = 1, n = 1, 2, ...\}$ the *p*- quasicyclic group, $Q_8 = < a, b; a^4 = b^4 = 1, a^{-1}ba = b^{-1} >$ the quaternion group, $C_G(A)$ the centralizer of *A* in *G*, $C(G)$ the center of *G*.

A group *G* is a solvable group, if it has the solvable derived series $G > G' > G^{(2)} > ... \ ... > G^{(n)} = <e>$. A group *G* is called Dedekind group if every subgroup *A* of *G* is a normal subgroup of *G*. Hamiltonian group is called a non-abelian Dedekind group. A group *G* is called a locally graded group if every finitely generated non-trivial subgroup of *G* contains a proper subgroup of finite index ([1] p. 236). The subgroup *A* is called quasicentral of the group *G* if each subgroup of *A* is the normal subgroup of *G*. If a group *G* is an extension of the group *H* by the normal subgroup *N* of *G*, then $G/N \cong H$. If a group *G* is an extension of the quasicyclic group by the finite group, then a group *G* is called an almost quasicyclic group.

Definition 1. An infinite non-abelian *G* is said to be an *IAN* group if there exists a subgroup *A* of *G* such that every infinite subgroup of *A* and every infinite subgroup of *G* contains *A* is a normal subgroup of *G*. The subgroup *A* is called a nodal subgroup. If *A* is an abelian subgroup, then *G* is *IANA* group.

Proposition 1. [[1], T. 6.10]. The class a solvable *INH* groups formed from the infinite Hamiltonian groups and the non-abelian non-Hamiltonian groups that are the finite extensions of the quasicyclic subgroups by the finite abelian and the finite Hamiltonian groups.

Proposition 2. [[7], T. 5.8]. Let *F* be a group and let $X = \{x_i; i \in I\} \neq$ Ǿ be a subset of *F* such that $< x_i >$ an infinite cyclic group for all x_i from *X*. If $F = \sum_{\in} < x_i >$ is the direct sum of the cyclic groups $< x_i >$ for every $i \in I$, then *F* is free abelian group with the set *X* of the free generators.

Lemma 1. If *G* is *IAN* group with a finite nodal subgroup A, then G/A is an abelian group or *INH* group.

Proof. If G/A is an abelian group, then Lemma 1 is valid. Let G/A be non-abelian groupand let B/A be an arbitrary infinite subgroup of G/A. Evidently, there exist $B \lhd G$ and $B/A \lhd G/A$. Thus G/A is *INH* group.

Lemma 2. If *G* is an infinite group with a finite commutator G' of the group *G*, then *G* is *IAN* group with a finite nodal subgroup G'.

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Proof.

If put $A = G'$, evidently, then G is *IAN* group with a finite nodal subgroup.

Lemma 3. Let G be *IAN* group with a nodal subgroup A. If a nodal subgroup A contains the elements of the infinite orders, then A is an abelian quasicentral subgroup of group G.

Proof. By Proposition 2 A is an abelian group. Let B be an arbitrary subgroup of the group A. If B is an infinite subgroup, then B is a normal subgroup of G.

Let B be a finite subgroup. By assumption, the group A contains the element x of the infinite order, A is an abelian group. If $B< x > \cong B \times< x >$, by definition *IAN groups* $B \times< x > \lhd G$ and furthermore $B \lhd G$. Thus A is an abelian quasicentral subgroup of group G. Lemma is proved.

Lemma 4. Let G be a locally graded *IAN* group with a nodal subgroup A. If there exists a subgroup A that is not a normal subgroup of G, then A is a finite group, or A is an extension of the quasicyclic subgroup by a finite Dedekind group.

Proof. Let G be *IAN* group with a nodal subgroup A, $A_1 < A$, A_1 is a non-normal subgroup in G. Evidently, $A_1 \neq < e >$, A_1 is a finite subgroup. By Lemma 3 A is a periodic group. If A is a finite group, then lemma is valid. Let A be an infinite periodic subgroup. We consider two possible cases: A is non ч – group, or A is ч – group.

Let A be non ч – group. Suppose that there exists the subgroup A_2 of A that $A_2 = A_3 \times A_4$, $A_2 \cap A_1 = < e >$ and A_3, A_4 are the infinite groups. By Definition 1 $A_3 \lhd G$, $A_3 \times A_1 \lhd G$, $A_4 \lhd G$, $A_4 \times A_1 \lhd G$. Evidently, $(A_3 \times A_1) \cap (A_4 \times A_1) = A_1$ and, furthermore, $A_1 \lhd G$. It is contradiction. Thus, A is not non ч – group.

Let A be ч – group. Denote $A = R.B$, R is the direct product of the finite number of the quasicyclic groups, R is a divisible group, B is the finite group, $B \neq < e >$. Therefore, A_1 is not a nonnormal subgroup of G. we assume that there exists a cyclic subgroup $< a >$ of A_1 non-normal in G, and also $R \cap < a > = < t >$. Since R is a divisible group there exists a quasicyclic subgroup R_1 of R and, furthermore, R_1 contains the subgroup $< t >$. Put $R = R_1 \times R_2$ where R_2 is an infinite subgroup of A, or $R_2 = < e >$. If R_2 is an infinite subgroup of A, by Definition 1 $R_2 \lhd G$ and, furthermore, $(R_2 \lambda < a >) \lhd G$, $R_1 \lhd G$, $(R_1 < a >) \lhd G$. Evidently, $(R_2 \lambda < a >) \cap (R_1 < a >) = < a >$ and $< a > \lhd G$. It is contradiction. Then $R_2 = < e >$, $R = R_1$ is quasicyclic group, and moreover, $A/R \cong B$ is a finite Dedekind group. Thus A is an extension of the quasicyclic subgroup by the finite Dedekind group. Lemma is proved.

Theorem 1. If G is a locally graded *IAN* group with a nodal subgroup A, then subgroup A belongs to one of the types:

1. A is a finite subgroup of G.
2. A is an extension of the quasicyclic subgroup by a finite Dedekind group, the commutator G' is an infinite group.
3. A is an infinite quasicentral periodic subgroup of G,
4. A is a quasicentral non-periodic abelian subgroup of G.

Proof. Let A be a non-quasicentral subgroup of G. By Lemma 4 the subgroup A of G belongs to one of type 1 or 2 of this theorem.

Let A be a quasicentral subgroup of G. By Lemma 3 the subgroup A of G belongs to one of type 3 or 4 of this theorem. Theorem is proved.

By Theorem 1 and definition *IANA* groups the next corollary follows.

Corollary 1. If G is a locally graded *IANA* group with a nodal subgroup A, then subgroup A belongs to one of the types:

1. A is a finite abelian subgroup of G.
2. $A = Z(p^{\infty}) \times B$, B is a finite group.
3. A is an infinite quasicentral periodic abelian subgroup of G.
4. A is a quasicentral nonperiodic abelian subgroup of G.

Theorem 2. G – *LAN* groups with a finite nodal subgroup are the groups belonging to one of the types:

1. G is an infinite group, G is an extension of the finite subgroup by Dedekind group,
2. G is an extension of the finite normal subgroup by an infinite non-solvable group whose every infinite subgroup is normal in G.
3. $G = R.H$, R is a quasicyclic group contained in G', H is a finite group contained in a normal subgroup N of G so that H/N is Dedekind group.

Proof. Let G be *IAN* group with a finite nodal subgroup A. If G/A is Dedekind group, then G is of type 1. If G/A is non-Dedekind group, by Lemma 1 G/A is non- Hamiltonian *INH* group. If G/A is a non-solvable group, then G is of type 2.

If G/A is a solvable group by Proposition 1 $G/A = (D/A).(H/A)$ where $D/A \lhd G/A$, D/A is a quasicyclic group, H/A is such a finite subgroup of G that the factor group $H/A/(D/A \cap H/A)$ is Dedekind group. Denote $G = D.H$, where $D \lhd G$, H is a finite group, and $A \leq D \cap H$. If G' is a finite group, then G/G' is an infinite group. By Lemma 2 G is a group of type 1.

Let G' be an infinite group. Assuming that D/A is a quasicyclic group, A is a finite group. Denote $D = A.C$ where $C = C_D(A)$ is a centralizer of A in D. Consequently, $C = (C_G(A) \cap D) \lhd G$, moreover $C \cap A \leq C(A)$. Thus, C is a central extension of the finite group by the quasicyclic group. Then D is an extension of the quasicyclic subgroup R by the finite group. Furthermore, $D = R.A$, $R \lhd G$ and $G = R.H$. Since G' is an infinite group, evidently $R \leq G'$. Let $N/A = D/A \cap H/A$ be the subgroup of G/A. By Isomorphism Theorem $(G/A)/(N/A) \cong G/N)$ $N \lhd G$ and H/N is Dedekind group. Thus G is a group of type 3.

Conversely, let G be the group belonging to one of types 1 to 3. We shall prove that G is group with a finite nodal subgroup. Let the group G be of type 1. Then G is an extension of the finite subgroup N by an infinite Dedekind group. Thus the commutator of the group G/N is a finite group, evidently the commutator of the group G is a finite group too. By Lemma 2 G is group with a finite nodal subgroup.

Let G be the group of type 2 or 3. Suppose that there exists a finite normal subgroup N of the group G so that G/N is non Dedekind group. Put $N = A$. Let B be an arbitrary infinite subgroup of G containing A. Then $B/A \lhd G/A$ implies $B \lhd G$. Since A is a finite group, evidently its every infinite subgroup is normal of G. Thus the groups of type 2 or 3 are *IAN* groups with a finite nodal subgroup. Theorem is proved.

Corollary 2. The locally graded *IAN* groups with a finite nodal subgroup are the groups to one of type 1 or 3 of Theorem 2.

Proof. If G is an locally graded *IAN* group with a finite nodal subgroup A, then G is the group to one of type 1 to 3 of Theorem 2. If G is the group to one of types 1 or 3 of Theorem 2, then Corollary 2 is valid. Prove that G is not the group of type 2 of Theorem 2. Let G be the group of type 2 of Theorem 2, then G/A is an infinite non-solvable *INH* group. If A is a finite group, and G locally graded group, evidently then G/A is a locally graded non-solvable *INH* group. It is contradiction. Thus G is not the group of type 2 of Theorem 2.

Corollary 3. The locally graded *IAN* group with a finite solvable nodal subgroup is the solvable group belonging to one of types 1 or 3 of Theorem 2.

Proof. Let G be an investigated group, A is its finite solvable subgroup. By Corollary 2 G/A is the group to one of types 1 or 3 of Theorem 2. Evidently, then G/A is the solvable group, by assumption A is the solvable subgroup. This implies that G is the solvable group.

Theorem 3. The locally graded *IANA* group with a finite nodal subgroup is the solvable group of degree less or equal to number 3.

Proof. If G/A and A are solvable groups, then G is the solvable group and $(G/A)'$ is abelian group. Using this fact it follows that $G^{(2)}$ is abelian group and $G^{(3)} = < e >$. Theorem is proved.

Theorem 4. The locally graded *IANA* groups with a finite nodal subgroup are the groups belonging to one of the types:

1. G is an extension of the finite normal abelian subgroup N by Dedekind group.
2. G is an almost quasicyclic group, G' is an almost quasicyclic abelian group.
3. $G = R.H$, R is a p- quasicyclic group, $R \lhd G$, H is a finite group containing a normal subgroup N of G so that H/N is Hamiltonian group, $R.N$ is abelian group, the commutator H' is non-abelian group.

Proof. Let G be locally graded *IANA* groups with a finite nodal subgroup. By Theorem 3 G is the solvable group. If G/A is abelian group, then G is the group of type 1 of this theorem.

Let G/A be non-abelian group. By Lemma 1 G/A is *INH* group. By Proposition 1 G/A is Hamiltonian group or an extension of the quasicyclic subgroup by the finite Dedekind group. If G/A is Hamiltonian group, then G is the group of type 1 of this theorem.

Let G/A be an extension of the quasicyclic subgroup B/A by the finite Dedekind group. We know that A is a finite abelian group, thus $|B : C_B(A)| < \infty$ and $A \leq C_B(A)$. If $A \leq C_B(A)$ and B/A is a quasicyclic group which does not contain the proper subgroup of the finite index, then $C_B(A) = B$ and thus $A \leq C_B$. It follows that B is a central extension of the finite abelian group A by the quasicyclic group (B is an almost quasicyclic group). Since R is a quasicyclic subgroup of B and a characteristic subgroup of B, thus R is a normal subgroup of G. By Isomorphism Theorem $(G/A)/(B/A) \cong G/B$ and furthermore G/B is finite Dedekind group. We consider two possible cases, G/B is an abelian group or Hamiltonian group.

Let G/B be an abelian group. Evidently, the commutator $G' \leq B$. If G' is a finite group, then G is the group of type 1 of this theorem.

If G' is an infinite group, evidently $G' \leq B$, then G and G' are almost quasicyclic groups. Thus G is the group of type 2 of this theorem.

Let G/B be Hamiltonian group. Evidently, G is an extension of the quasicyclic group R by the finite group. Denoting $G = R.H$, H is a finite group, consequently $N = B \cap H$, $B = R.N$, this implies $G/B \cong H/N$. Thus H/N is finite Hamiltonian group.

If G' is a finite group, then G is the group of type 1 of this theorem. If G' is an infinite group, then G is the group of type 2 of this theorem.

Let G' be an infinite non-abelian group. Put $G = R.H$, where R is a normal subgroup of G. It is well- know that $C_G(R) \geq$ G, $G' = R'.H'$. $[R,H] \leq R.H'$, and $[R.H'] = < e >$. From the product of the groups M and N, where M is the central group and N is abelian group and $G' \leq R.H'$, it follows that H' is nonabelian group. Thus G is the group of type 3 of the Theorem 4.

Conversely, let G be an infinite group one of types 1 to 3. If G is one of type 1 or 3, then we put $A = N$. If G is of type 2 we put $G' = R \times D$ which is the direct product of a quasicyclic group R and a finite group D. If A is the subgroup of G generated by the elements of G' whose orders are divisors of the order of D, then A is abelian group and $D \leq A$.

Let G be the group of type 1. Then G/A is Dedekind, there exits B/A an arbitrary normal subgroup of G/A. Consequently, $B \lhd G$.

Let G be the group of type 2. We have $G' = R \times D = R.A$. It follows that G/A is an extension of the quasicyclic group $G'/A = = R.A/A$ by the finite abelian group. Thus B/A is an arbitrary normal subgroup in G/A. Consequently, $B \lhd G$.

Let G be the group of type 3. In that case G/A is an extension of the quasicyclic group $R.A/A$ by the finite Hamiltonian group

H/A. Evidently, $B/A \lhd G$ implies $B \lhd G$. If G is the group one of types 1 to 3 this theorem, then G is *IANA* group with a finite nodal subgroup A. Theorem is proved.

Remark. The class of the infinite solvable *IANA* groups contains the class of solvable *INH* and the class *KI* groups with the finite commutator.

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