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COORDINATION OF BUS DEPARTURES BY MATHEMATICAL PROGRAMMING

This paper deals with coordination of bus arrivals. A quality criterion of this problem is to minimize waiting time of passengers, to make public transport more attractive. One possibility for solving is to increase the number of arrivals with associated investments. Another possibility, without investment, is the coordination of bus arrivals. The latter possibility is analyzed and solved in this paper. Mathematical formulation of this problem leads to a quadratic programming model which is hard to solve. Our approach is based on piecewise linearization of the quadratic objective function. This integer programming model enables to include to the problem some other non-trivial aspects of arrival coordination. These additional aspects are the necessity of safety break compliance and order rearrangement of bus arrivals at a given bus stop. In this paper, integer programming models of the above mentioned problems are presented and the associated numerical experiments are reported to enable comparison of the suggested approaches.

Keywords: *Public transport, coordination of bus arrivals, waiting time of passengers, integer programming, free order of objects.*

1. Introduction

A mathematical programming model of the regular arrival deployment problem was formulated and solved in the several last decades to obtain solution of time coordination of bus arrivals. In this model, regular distribution of time intervals was taken as a quality criterion of searched solution, even if the original objective was to minimize the total waiting time of passengers. Works with time coordination of bus arrivals were published in [1], [2], [3] and [4]. We focused other approach on the non-investment increasing of public transport attractiveness.

In the problem, n arrivals of vehicles at a stop in the designate headway are considered. Let t_i be arrival time of vehicle i at the stop. The earliest possible arrival time of the vehicle i is denoted as ai. This time may be postponed at most until the time $a_i + c_i$, where c_i is the maximum possible shift of arrival at the stop. It is necessary to find such time positions of the individual arrivals so that the total passengers' waiting time is minimal.

Arrival times t_0 and tn are fixed. The goal is to shift times t_i for $i = 1, ..., n - 1$, so that the overall waiting time of passengers in passenger-minutes is minimal. Fig. 1 shows how the waiting time depends on the arrival time distribution in the time headway $\langle t_0, t_n \rangle$. The grey area represents the total waiting time of passengers coming to the stop in a given headway and waiting for a bus.

The total waiting time of passengers in the headway $\langle t_0, t_n \rangle$ can be expressed as:

$$\sum_{i=1}^{n} \frac{1}{2} f (t_i - t_{i-1})^2 = \frac{1}{2} f \sum_{i=1}^{n} v_i^2 \tag{1}$$

where we introduce a variable v_i, which represents the length of time interval between two succeeding arrivals t_{i-1} and t_i for $i = 1, ..., n$ on the assumption that passengers arrive at the stop equally, uniformly with intensity f, where we assume that f.

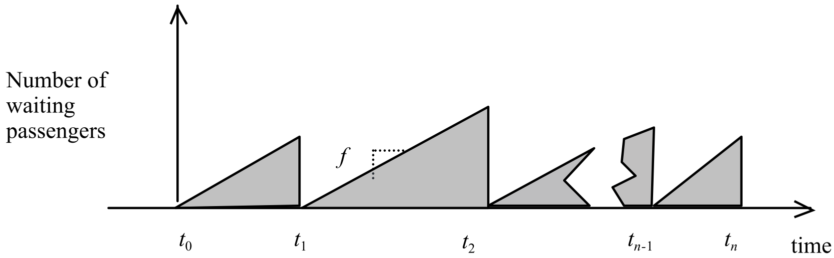

Fig. 1 The waiting time of passengers during headway $\langle t_0, t_n \rangle$

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We consider here the situation when the stop is serviced by busses, which follow different routes, but a considerable part of their routes is common. We also assume that demanded arrival times, which are most suitable for the passengers using only the common part, are evenly distributed. Then even the passengers, which arrive at the stop accordingly to a time table, loose their time either out of the stop or at the ends of their trips.

2. Mathematical model of the basic problem

We introduce variable x_i for $i = 1, ..., n - 1$, which models a shift of the arrival time t_i from the earliest possible arrival time a_i. Then the mathematical model of the problem will be as follows:

$$\text{Minimize} \quad \frac{1}{2} f \sum_{i=1}^{n} v_i^2 \tag{2}$$

$$\text{Subject to} \quad x_1 + a_1 - t_0 = v_1 \tag{3}$$

$$x_1 + a_i - x_{i-1} - a_{i-1} = v_i \quad \text{for } i = 2, ..., n - 1 \tag{4}$$

$$t_n - x_{n-1} - a_{n-1} = v_n \tag{5}$$

$$v_i \geq 0 \quad \text{for} \quad i = 1, ..., n \tag{6}$$

$$x_i \leq c_i \quad \text{for} \quad i = 1, ..., n - 1 \tag{7}$$

$$x_i \geq 0 \quad \text{for} \quad i = 1, ..., n \tag{8}$$

The constraints (3) - (5) determine the gap between two consecutive arrivals as the relation between the values of the variables v_i and x_i. Due to the squares of variables v_i, $i = 1, ..., n$, the objective function (2) is non-linear. In [5] and [6], the similar situation was solved using rearrangement of the original non-linear problem to simpler max-min problem. The following research was focused on search for other approaches, which were published in [7], [8], [9] and [10] together with numerical experiments on real data.

In this paper we will approximate the quadratic non-linear function (2) by a piecewise linear function. This linearization approach has been analyzed in details in [11] and [12].

3. Arrival deployment problem with the constraints for compliance of safety breaks

3.1 Mathematical model of arrival deployment problem

This section is focused on real problem instances of public transport with the constraints for compliance of safety breaks. The safety breaks are important elements of each vehicle schedule. The safety breaks serve for the relaxation of drivers. It is necessary to place them into the schedule and to take into account that they must not be interrupted. The situation is depicted in Fig. 2.

There are two alternatives (time windows) for the possible arrival time of the bus arrival i: either $t_i \in \langle a_i, a_i + c_i \rangle$ or $t_i \in \langle a_i + c_i + s_i, a_i + r_i \rangle$, where si is the length of the safety break and r_i is the maximal shift of the arrival time t_i from a_i in the second time window. We introduce these constraints for the compliance of the safety breaks into the mathematical model (2) - (8). As for the objective function, we use a linearization approach for linearization of (2).

The mathematical model of the problem with the constraints for compliance of the safety breaks will be as follows:

$$\text{Minimize} \quad \frac{1}{2} f \sum_{i=1}^{n} \sum_{j=1}^{m(i)} (2j - 1) \cdot u_{ij} \tag{9}$$

$$\text{Subject to} \quad x_1 + a_1 - t_0 = \sum_{j=1}^{m(1)} u_{1j} \tag{10}$$

$$x_i + a_i - x_{i-1} - a_{i-1} = \sum_{j=1}^{m(i)} u_{ij} \quad \text{for} \quad i = 2, ..., n - 1 \tag{11}$$

$$t_n - x_{n-1} - a_{n-1} = \sum_{j=1}^{m(n)} u_{nj} \tag{12}$$

$$x_i \geq (c_i + s_i) y_i \quad \text{for} \quad i = 1, ..., n - 1 \tag{13}$$

$$x_i \leq c_i + (r_i - s_i) y_i \quad \text{for} \quad i = 1, ..., n - 1 \tag{14}$$

$$x_i \geq 0 \quad \text{for} \quad i = 1, ..., n - 1 \tag{15}$$

$$u_{ij} \geq 0 \quad \text{for} \quad i = 1, ..., n, j = 1, ..., m(i) \tag{16}$$

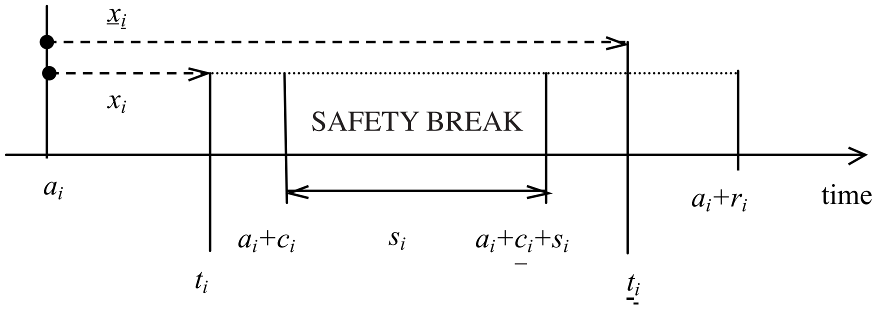

Fig. 2 The possible arrival times ti and ti in regard to position of the safety break

$$u_{ij} \leq 0 \quad \text{for} \quad i = 1, ..., n, j = 1, ..., m(i) \tag{17}$$

$$y_i \in \{0,1\} \quad \text{for} \quad i = 1, ..., n \tag{18}$$

We approximate the quadratic function (2) by a piecewise linear function (9) by introducing new variables u_{ij} for $j = 1, ..., m(i)$, $i = 1, ..., n$ subject to $\sum_{j=1}^{m(i)} u_{ij} = v_i$ for $i = 1, ..., n$.

The constant $m(i)$ for $i = 1, ..., n$ denotes the number of dividing points of the quadratic function approximation. We consider that each of the time intervals is divided into single minutes. If all the time constants are given in minutes, then we can model these constants as:

$$m(1) = a_1 + r_1 - t_0$$

$$m(i) = a_i + r_i - a_{i-1} \quad \text{for} \quad i = 2, ..., n - 1$$

$$m(n) = t_n - a_{n-1}.$$

The constraints (13) – (14) ensure the compliance of safety breaks. For this purpose, we introduce zero-one variables y_i to the constraint so that either $x_i \leq r_i$ and $x_i \geq c_i + s_i$ for $y_i = 1$ or $x_i \leq c_i$ and $x_i \geq 0$ for $y_i = 0$ hold.

Note, that even if model (9) – (18) contains more constraints than model (2) – (8), the set of feasible solutions of the problem (9) – (18) is bigger than the solution set of the basic problem.

3.2 Numerical experiments with the constraints for compliance of safety breaks

As the optimal solution does not depend on a concrete value of the constant f, we put $f = 1$ passenger per minute in the following experiments.

We present here a series of numerical experiments by IP-Solver for 8 real problems of the public transport in the selected area of the Czech Republic. In Table 1 we show the comparison of the results of linearized basic problem (2) – (8) and the problem (9) – (18) with the constraints for the compliance of the safety breaks. The benchmark criterion for the comparison is the total waiting time of passengers in [passenger-minutes], which is the value of the objective function (2). The "default state" mentioned in Table 1 denotes the current deployment of arrivals the running transportation system.

The row "The safety breaks Offered/Used" denotes number of bus arrivals which can use the system of two time windows (separated by a safety break, which is depicted in Fig. 2). The value (used) is number of bus arrivals, which were situated into the second time window $t_i \in \langle a_i + c_i + s_i, a_i + r_i \rangle$. Usage of the second time window is followed by better value of the objective function.

4. Problem of time coordination with free order of arrivals at a single stop

This section is focused on time coordination with a fixed order of bus arrivals. Now, we deal with the problem of time coordination with a free order of bus arrivals. On the contrary to the previous problems in this case a rearrangement of order of bus arrivals at a given bus stop is possible. The associated research is published in [13] together with results of preliminary numerical experiments on real data.

The coordination of arrival times t_i and t_k are given by two alternatives for the possible order of the bus arrival i and k. The examples are illustrated in Figs. 3 and 4.

4.1 Mathematical model of time coordination with a free order of arrivals

We introduce zero-one variables wik for $i = 0, ..., n - 1, k = 1, ..., n$, which model whether the bus arrival i directly precedes the bus arrival k or not. These variables are defined only for pairs (i, k) of bus arrivals where the direct preceding is possible. Thus

w_{0k} is defined for $k = 1, ..., n - 1$,
because the fixed arrival 0 directly precedes each free arrival k,
win is defined for $i = 1, ..., n - 1$, (19)

Comparison of results of the linearized basic problem (2) – (8) and the problem (9) – (18) with the constraints for the compliance of the safety breaks Table 1

The code of problem	1	2	3	4	5	6	7	8
Number of arrivals	15	19	12	9	9	9	23	9
The safety breaks Offered/Used	3/0	3/1	4/0	2/2	3/3	2/1	5/0	2/2
The variants of solution	The total waiting time of passengers [passenger-minutes]							
The default state	5508	6345.5	5628	9047.5	5250.5	5618	5460	5166
The linearized basic problem	5125	5319.5	4327	7837.5	4893.5	4694	4975	4464
The problem with the constraints for the compliance of the safety breaks	5125	4776.5	4327	6678.5	3812.5	4429	4975	3861

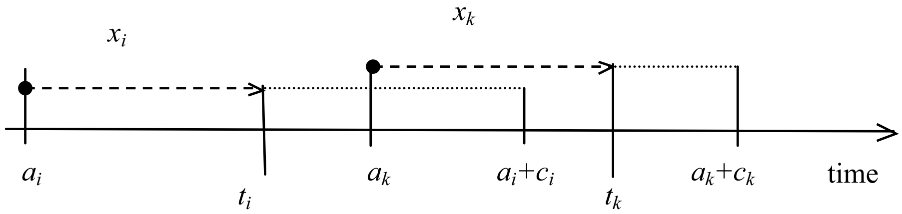

Fig. 3 The possible arrival times t_i and t_k, when ti precedes t_k

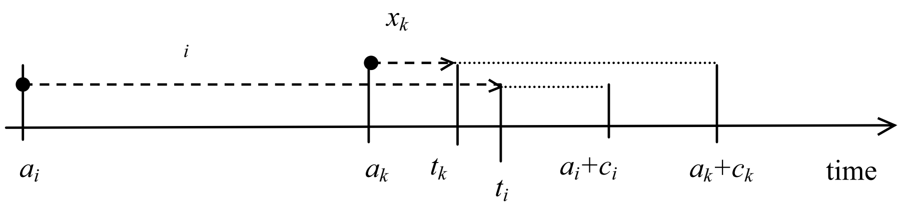

Fig. 4 The possible arrival times t_i and t_k, when t_k precedes t_i

because bus arrival i directly precedes the fixed arrival n, w_{ik} is defined for $i = 1, ..., n-1$, $k = 1, ..., n-1$, $i \neq k$ subject to $a_i < a_k + c_k$.

To make the following model more concise, we introduce a Boolean function *exists* (w_{ik}) according to (19) which takes the value of true if a variable w_{ik} is defined and the value of false otherwise. Then the mathematical model of the problem with a free order of arrivals will be as follows:

$$\text{Minimize} \quad \frac{1}{2} f \sum_{i=1}^{n} \sum_{j=1}^{m(i)} (2j-1) \cdot u_{ij} \tag{20}$$

$$\text{Subject to} \quad x_k + a_k - t_0 \leq \sum_{j=1}^{m(1)} u_{k,j} + (t_n - t_0) \cdot (1 - w_{0k})$$

for $k = 1, ..., n-1$ (21)

$$x_k + a_k - t_0 \geq \sum_{j=1}^{m(1)} u_{k,j} - (t_n - t_0) \cdot (1 - w_{0k})$$

for $k = 1, ..., n-1$ (22)

$$x_k + a_k - x_i - a_i \leq \sum_{j=1}^{m(1)} u_{k,j} + (t_n - t_0) \cdot (1 - w_{ik})$$

for $i = 1, ..., n-1$; $k = 1, ..., n-1$; *exist*(w_{ik}) (23)

$$x_k + a_k - x_i - a_i \geq \sum_{j=1}^{m(1)} u_{k,j} - (t_n - t_0) \cdot (1 - w_{ik})$$

for $i = 1, ..., n-1$; $k = 1, ..., n-1$; *exist*(w_{ik}) (24)

$$t_n - x_i - a_i \leq \sum_{j=1}^{m(n)} u_{n,j} + (t_n - t_0) \cdot (1 - w_{in})$$

for $i = 1, ..., n-1$ (25)

$$t_n - x_i - a_i \geq \sum_{j=1}^{m(n)} u_{n,j} - (t_n - t_0) \cdot (1 - w_{in})$$

for $i = 1, ..., n-1$ (26)

$$\sum_{i=0}^{n-1} w_{ik} = 1 \quad \text{for } k = 1, ..., n; \ exist(w_{ik}) \tag{27}$$

$$\sum_{k=1}^{n} w_{ik} = 1 \quad \text{for } i = 0, ..., n-1; \ exist(w_{ik}) \tag{28}$$

$$x_i \geq 0 \quad \text{for } i = 1, ..., n-1 \tag{29}$$

$$x_i \leq c_i \quad \text{for } i = 1, ..., n-1 \tag{30}$$

$$u_{ij} \geq 0 \quad \text{for } i = 1, ..., n; j = 1, ..., m(i) \tag{31}$$

$$u_{ij} \leq 1 \quad \text{for } i = 1, ..., n; j = 1, ..., m(i) \tag{32}$$

$w_{ik} \in \{0,1\}$ for $i = 1, ..., n-1$; $k = 1, ..., n-1$;

exist(w_{ik}) (33)

We approximate the quadratic function (2) by a piecewise linear function (20) as mentioned in section 3. In this case the constant $m(i)$ for $i = 1, ..., n$ denotes the number of dividing points of the approximation of the quadratic function (2). We consider that each of the time intervals is also divided into single minutes. Then we can model these constants as:

$m(1) = a_1 + c_1 - t_0,$
$m(i) = a_i + c_i - t_0$ for $i = 2, ..., n - 1,$
$m(n) = t_n - t_0.$

The constraints (21) - (26) cause that if $w_{ik} = 1$ for some pair (i, k), then the gap between two succeeding arrivals t_i and t_k is equal to the sum of u_{kj} for $j = 1, ..., m(i)$, which models the time between two succeeding arrivals. If $w_{ik} = 0$ holds, then the associated constraints are relaxed by suitable value of $t_n - t_0$. The constraints (27) assure that exactly one arrival i precedes arrival k. The constraints (28) assure that exactly one arrival i precedes arrival k. The output of the problem (20) - (33) solving is formed by $n+1$ values w_{ik} for $w_{ik} = 1$, which define a new rearrangement of the order of arrivals.

4.2 Numerical experiments for time coordination with a free order of bus arrivals

We present here a series of numerical experiments performed on IP-Solver for 16 real problems of the public transport in the selected area of the Czech Republic. In Tables 2 and 3, we show the comparison of results of the linearized basic problem (2) - (8) and the problem (20) - (33) with a free order of arrivals. The benchmark criterion for the comparison is the total waiting time of passengers in [passenger-minutes], which is the value of the objective function (2).

In Table 4 we show the comparison of results of one example - the code 8 of the public transport in the area of Frydek Mistek - Dobra for 9 arrivals. We show output of the gap between two arrivals in [minutes] and the total waiting time of passengers in [passenger-minutes] for the linearized basic problem (2) - (8) and the problem (20) - (33) with a free order of arrivals. In this case, the result of the problem with a free order of bus arrivals designs a new rearrangement of the order of arrivals: 0, 2, 1, 4, 3, 5, 6, 7, 8, 9.

5. Conclusion

The paper was focused on the non-investment increasing of public transport attractiveness. The mathematical model with constraints for the compliance of safety breaks and mathematical model

Comparison of results of the linearized basic problem (2) - (8) and the problem (20) - (33) with free order of arrivals Table 2

The code of problem	1	2	3	4	5	6	7	8
Number of arrivals	15	19	12	9	9	9	23	9
The variants of solution	The total waiting time of passengers [passenger-minutes]							
The default state	5508	6345.5	5628	9047.5	5250.5	5618	5460	5166
The linearized basic problem	5125	5319.5	4327	7837.5	4893.5	4694	4975	4464
The problem with free order of arrival	5036	5319.5	4326	7837.5	4731.5	4694	4961	4191

Comparison of results of the linearized basic problem (2) - (8) and the problem (20) - (33)with free order of arrivals Table 3

The code of problem	9	10	11	12	13	14	15	16
Number of arrivals	10	9	10	10	9	5	10	9
The variants of solution	The total waiting time of passengers [passenger-minutes]							
The default state	4536	6634	9923.5	9529	8282	21711.5	8007	2131
The linearized basic problem	3557	4169	8945.5	6525	5837	18029.5	6930	2084
The problem with free order of arrival	3555	3932	7144.5	6424	5835	18029.5	6872	2084

Comparison of results of the linearized basic problem (2) - (8) and the problem (20) - (33) with free order of arrivals for one problem 8 Table 4

Problem 8	The gap between two arrivals [minutes]									The total waiting time of passengers [passenger-minutes]
The variants of solution	$t_1 - t_0$	$t_2 - t_1$	$t_3 - t_2$	$t_4 - t_3$	$t_5 - t_4$	$t_6 - t_5$	$t_7 - t_6$	$t_8 - t_7$	$t_9 - t_8$	
The default state	13	13	45	10	65	18	22	30	44	5166
The linearized basic problem	13	13	43	12	46	30	29	36	38	4464
The problem with free order of arrival	25	8	38	18	38	29	30	36	38	4191

with a free order of arrivals are presented and compared to the basic problem.

The results of the real problems with models (9) – (18) with the constraints for compliance of safety breaks shown in Table 1 are better than the results obtained with the linearized basic problem (2) – (8).

Also the results of real problems (20) – (33) with a free order of arrivals shown in Tables 2 and 3 are better comparing to the linearized basic problem (2) – (8).

Taking into account that all the computational time for the solved problems did not exceed few seconds, we can conclude that the higher complexity of used models brought considerable savings of passengers' waiting time.

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