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ON FREQUENCY ESTIMATION FOR PARTIALLY OBSERVED SYSTEM WITH SMALL NOISES IN STATE AND OBSERVATION EQUATIONS

We consider the problem of frequency estimation of the periodic signal multiplied by a Gaussian process (Ornstein-Uhlenbeck) and observed in the presence of the white Gaussian noise. We demonstrate the consistency and asymptotic normality of the maximum likelihood and Bayesian estimators in the sense of the small noise asymptotics. The model of observations is a linear nonhomogeneous partially observed system and the construction of the estimators is based on the Kalman-Bucy filtration equations. For the study of the properties of the estimators, we apply the techniques introduced by Ibragimov and Has'minskii.

Keywords: *partially observed linear system, stochastic signal, frequency estimator, maximum likelihood method, Bayesian approach, characteristics of estimators, small noise asymptotic*

1. Introduction

The problem of the estimation of the frequency (Doppler shift) of the signal is of a great importance in the fields of radio-and-hydroacoustic communications and positioning, radio-and-hydrolocation, radio-and-hydroacoustic positioning, etc. [1-3]. The simplest model for the useful signal is the harmonic wave with the amplitude A and the frequency ϑ_0:

$$s(t) = A\cos(2\pi\vartheta_0 t) \qquad (1)$$

While propagating, the signal $s(t)$ can be distorted by both additive $n(t)$ and multiplicative $Y(t)$ interferences [3-6]. As the model of the additive interferences, following [1-3], we choose the white Gaussian noise (WGN) with the double-sided spectral density σ_2 (correlation function $\langle n(t_1)n(t_2)\rangle = \sigma^2\delta(t_2 - t_1)$). For the statistical description of the multiplicative interferences, the stationary centered Gaussian random process can be often applied [3, 7]. As a result, the mix

$$X(t) = A\cos(2\pi\vartheta_0 t)Y(t) + n(t), \quad 0 \le t \le T \qquad (2)$$

is available for an observation.

We presuppose that the process $Y(t)$ satisfies a linear stochastic equation. The simplest examples are the following.

Example 1. Ornstein-Uhlenbeck (O-U) process

$$\dot{Y}(t) + aY(t) = bN(t), \quad 0 \le t \le T \qquad (3)$$

where $\dot{Y}(t) = dY(t)/dt$; $a > 0$, $b > 0$ are certain parameters; $N(t)$ is WGN with the unit double-sided spectral density, while the processes $n(t)$ and $N(t)$ are statistically independent. As it is well known [2], the correlation function $R(\tau)$ and the spectral density $G(\lambda)$ of the process $Y(t)$ are determined as

$$R(\tau) = b^2\exp(-a|\tau|)/2a \quad G(\lambda) = b^2/(a^2 + 4\pi^2\lambda^2) \quad (4)$$

Example 2. The process $Y(t)$ described by the equation

$$\ddot{Y}(t) + a\dot{Y}(t) + a_0 Y(t) = bN(t), \quad 0 \le t \le T \qquad (5)$$

subject to condition $a_1^2 < 4a_0$. For the correlation function and the spectral density of the process $Y(t)$ (5), we have [2]

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$$R(\tau) = \frac{b^2}{2a_0a_1} \exp(-\gamma|t|)\left[\cos(\omega\tau) + \frac{\gamma}{\omega}\sin(\omega|\tau|)\right]$$
$$G(\lambda) = \frac{b^2}{[a_0 - 4\pi^2\lambda^2]^2 + 4\pi^2 a_1^2\lambda^2} \tag{6}$$

Here $\gamma = a_1/2$ and $\gamma^2 + \omega^2 = a_0$.

For simplicity of description, we presuppose that $Y(t)$ is the O-U process and we refer to the formalism of stochastic calculus. It should be noted that all the results presented below for the O-U process can be directly extended to encompass not only the model presented in Example 2, but the much more complicated linear partially observed systems as well, though in this case the calculus and the expressions involved turn to be much more cumbersome.

Thus, we have the observation model (2) and (3) presented in the form of the two-dimensional stochastic process $(X_t, Y_t, 0 \le t \le T)$ satisfying the differential equations

$$dX_t = A\cos(2\pi\vartheta_0 t)Y_t dt + \sigma dW, \quad X_0 = 0$$
$$dY_t = -aY_t dt + bdV_t, \qquad\qquad Y_0 = y_0 \tag{7}$$

Here $W_t, 0 \le t \le T$ and $V_t, 0 \le t \le T$ are two independent Wiener processes. The parameters A, σ, a, b are presupposed to be already known and positive, while the parameter $\vartheta_0 \in \Theta = (a,\beta)$ is unknown and has to be estimated by the observations $X^T = X(T)$ (2). We have partially observed linear system, where the component Y_T is not observable.

In [1, 3], the estimate is considered of the band center of random process with the fractional rational spectrum observed against Gaussian white noise; in [5], the results [1, 3] are generalized for the case when the spectral density of the random process is described by Gaussian curve. At last, in [6], the characteristics are found of the estimate of the band center of random process with free-form spectral density. And it is presupposed everywhere that the analyzed random process is stationary, while the asymptotic properties of the introduced band center estimates are examined under $T \to \infty$ only. The further extension of the specified problem is the use of non-stationary model of the useful random signal of the type (7) describing a practically important wide class of stochastic information processes and obtaining the properties of the estimate of unknown frequency parameter in wider range of asymptotics. In such context, the problem of statistical analysis of the estimate of random process band center is considered for the first time.

We now study two estimators of the parameter ϑ: $\hat\vartheta_T$ – the maximum likelihood estimator (MLE) and $\tilde\vartheta_T$ – the Bayesian estimator (BE). Let us remind the definitions of these estimators. We designate the current value of the unknown parameter ϑ_0 as ϑ and introduce the conditional mathematical expectation

$$m(\vartheta, t) = E_\vartheta(Y_t|X_s, 0 \le s \le t), \quad 0 \le t \le T \tag{8}$$

Then, according to [8], the likelihood ratio function can be formulated as

$$V(\vartheta, X^T) = \exp\left[\frac{1}{\sigma^2}\int_0^T \cos(2\pi\vartheta t)m(\vartheta, t)dX_t - \right.$$
$$\left. -\frac{1}{2\sigma^2}\int_0^T \cos^2(2\pi\vartheta t)m^2(\vartheta, t)dt\right], \quad \vartheta \in \Theta \tag{9}$$

The MLE $\hat\vartheta_T$ is determined by the relation [1-3]

$$V(\hat\vartheta_T, X^T) = \sup_{\vartheta \in \Theta} V(\vartheta, X^T) \tag{10}$$

To introduce the BE $\tilde\vartheta_T$, we presuppose that the parameter ϑ is a random variable having the continuous positive probability density $p(\theta)$, with $\alpha < \theta < \beta$ and the quadratic loss function. Then the BE is [1-3]

$$\tilde\vartheta_T = \int_\alpha^\beta \theta p(\theta)V(\theta, X^T)d\theta \bigg/ \int_\alpha^\beta p(\theta)V(\theta, X^T)d\theta \tag{11}$$

In our research, we are interested in the asymptotic behavior of these estimators. The asymptotics providing the consistency of the estimators can refer to the following groups of conditions: a) $\sigma \to 0$, $b \to 0$, T is fixed; b) $\sigma \to 0$, b and T are fixed; c) $T \to \infty$, σ and b are fixed. One should remember that the asymptotics $A \to \infty$ can be reduced to the case b).

In all the three cases, the problem of the parameter estimation is a regular one, therefore, the MLE and the BE are asymptotically normal with the natural normalization by the Fisher information $I_T(\vartheta_0)$ [1-3]:

$$\sqrt{I_T(\vartheta_0)}(\hat\vartheta_T - \vartheta_0) \Rightarrow N(0,1)$$
$$\sqrt{I_T(\vartheta_0)}(\tilde\vartheta_T - \vartheta_0) \Rightarrow N(0,1) \tag{12}$$

According to [8], for the Fisher information in Equation (12) we can write

$$I_T(\vartheta) = \frac{A^2}{\sigma^2}\int_0^T [\cos(2\pi\vartheta t)\dot m(\vartheta, t) - $$
$$-2\pi t\sin(2\pi\vartheta t)m(\vartheta, t)]^2 dt \tag{13}$$

Here and in the sequel dot means differentiation w.r.t. ϑ and prime differentiation w.r.t. t.

We suggest a slightly more general model for the observations

$$dX_t = f(\vartheta_0 t)Y_t dt + \sigma dW_t, \quad X_0 = 0, \quad 0 \le t \le T \tag{14}$$

$$dY_t = -a(t)Y_t dt + b(t)dV_t, \quad Y_0 = y_0 \tag{15}$$

where $f(t)$ is a known smooth periodic function with the period 1, and for the asymptotics we take the case a) when there is a small noise asymptotics in the state (15) and the observation (14) equations:

$$\sigma = b = \varepsilon \to 0 \tag{16}$$

The case b) is considered in the paper [9]. The statistical problems for the partially observed linear and nonlinear systems are studied in [10]. You can find there useful references to the related statistical problems. The case $T \to \infty$ can be treated following the study of similar problems in [11], where identification of the partially observed system is considered.

2. Auxiliary results

The process $m(\vartheta, t)$, $0 \le t \le T$ satisfies the following equation of Kalman-Bucy filtration [8, 12]:

$$dm(\vartheta, t) = -a(t)m(\vartheta, t)dt + \gamma(\vartheta, t)f(\vartheta, t) \cdot$$
$$\cdot [dX_t - f(\vartheta t)m(\vartheta, t)dt]/\sigma^2 \qquad (17)$$

where the function $\gamma(\vartheta, t) = \mathbf{E}_\vartheta(m(\vartheta, t) - Y_t)^2$ is the solution for the Ricatti equation.

$$\frac{\partial \gamma(\vartheta, t)}{\partial t} = -2a(t)\gamma(\vartheta, t) -$$
$$-\frac{1}{\sigma^2}\gamma^2(\vartheta, t)f^2(\vartheta t) + b^2(t), \qquad (18)$$
$$\gamma(\vartheta, t) = 0$$

Then, for the derivate $\dot{m}(\vartheta, t)$, we obtain

$$d\dot{m}(\vartheta, t) = -[a(t) + \gamma(\vartheta, t)f^2(\vartheta t)/\sigma^2]\dot{m}(\vartheta, t)dt +$$
$$+ h(\vartheta, t)m(\vartheta, t)dt + g(\vartheta, t)[dX_t - f(\vartheta t)m(\vartheta, t)dt] \qquad (19)$$

where $\dot{m}(\vartheta, 0) = 0$ and

$$g(\vartheta, t) = [\dot{\gamma}(\vartheta, t)f(\vartheta t) + t\gamma(\vartheta, t)f'(\vartheta, t)]/\sigma^2$$
$$h(\vartheta, t) = t\gamma(\vartheta, t)f(\vartheta t)f'(\vartheta, t)/\sigma^2 \qquad (20)$$

For the derivate $\dot{\gamma}(\vartheta, t)$, from Equation (18), we get the equation

$$\frac{\partial \dot{\gamma}(\vartheta, t)}{\partial t} = -2\left[a(t) + \frac{1}{\sigma^2}\gamma(\vartheta, t)f^2(\vartheta t)\right] \cdot$$
$$\cdot \dot{\gamma}(\vartheta, t) - \frac{2t}{\sigma^2}\gamma^2(\vartheta, t)f(\vartheta t)f'(\vartheta t), \ \dot{\gamma}(\vartheta, 0) = 0 \qquad (21)$$

These equations are obtained by the formal differentiation but this derivation can be justified by the standard methods. The both functions (the Gaussian – $m(\vartheta, t)$ and the deterministic – $g(\vartheta, t)$) are infinitely differentiable.

Equations (19) and (21) are linear and their solutions can be found explicitly. Let us denote

$$q(\vartheta, t) = a(t) + \gamma(\vartheta, t)f^2(\vartheta t)/\sigma^2 \qquad (22)$$

Then we get

$$\dot{m}(\vartheta, t) = \int_0^t \exp\left[-\int_s^t q(\vartheta, v)dv\right]h(\vartheta, s)m(\vartheta, s)ds +$$
$$+ \int_0^t \exp\left[-\int_s^t q(\vartheta, v)dv\right]g(\vartheta, s)[dX_s - f(\vartheta s)m(\vartheta, s)ds] \qquad (23)$$

and

$$\dot{\gamma}(\vartheta, t) = \frac{1}{\sigma^2}\int_0^t s\exp\left[-2\int_s^t q(\vartheta, v)dv\right] \cdot$$
$$\cdot \gamma^2(\vartheta, s)f'(\vartheta s)ds \qquad (24)$$

3. Main result

While analyzing the specified asymptotics (16), we denote the MLE and the BE as $\hat{\vartheta}_\varepsilon$ and $\tilde{\vartheta}_\varepsilon$, correspondingly and we present the model of observations as follows

$$dX_t = f(\vartheta_0 t)Y_t dt + \varepsilon dW_t, \quad X_0 = 0, \quad 0 \le t \le T \qquad (25)$$

$$dY_t = -a(t)Y_t dt + b(t)\psi(\varepsilon)dV_t, \quad Y_0 = y_0 \ne 0 \qquad (26)$$

Our goal is to estimate the parameter ϑ_0 and to describe the properties of the estimators for the case when $\varepsilon \to 0$, $\psi(\varepsilon) \to 0$. We presuppose that the functions $a(t)$, $b(t)$ are bounded and positive. In this study we also assume that

$$\psi(\varepsilon) = \mu\varepsilon \qquad (27)$$

The case $\psi(\varepsilon) \to 0$, $\psi(\varepsilon)/\varepsilon \to \infty$ and the case $\psi(\varepsilon) \to 0$, $\psi(\varepsilon)/\varepsilon \to 0$ will be considered later, using the same method applied.

We introduce the function $\gamma_*(\vartheta, t) = \varepsilon^{-2}\gamma(\vartheta, t)$ and point out that

$$dm(\vartheta, t) = -a(t)m(\vartheta, t)dt + \gamma_*(\vartheta, t)f(\vartheta t) \cdot$$
$$\cdot [dX_t - f(\vartheta t)m(\vartheta, t)dt] \qquad (28)$$

If the Equation (27) holds, then the function $\gamma_*(\vartheta, t)$ satisfies the Riccati equation

$$\frac{\partial \gamma_*(\vartheta, t)}{\partial t} = -2a(t)\gamma_*(\vartheta, t) - \gamma_*^2(\vartheta, t)f^2(\vartheta t) +$$
$$+ \mu^2 b^2(t), \quad \gamma_*(\vartheta, t) = 0 \qquad (29)$$

and the solution $\gamma_*(\vartheta, t)$ does not depend on ε.

We can show now that the Fisher information in this problem is

$$I_\varepsilon(\vartheta) = \varepsilon^{-2} I_0(\vartheta)(1 + o(1)) \tag{30}$$

where

$$I_0(\vartheta) = \int_0^T \left[f(\vartheta t)\dot{m}_0(\vartheta,t) + tf'(\vartheta t)m_0(\vartheta,t) \right]^2 dt \tag{31}$$

and $m_0(\vartheta,t), \dot{m}_0(\vartheta,t)$ are the limits of the processes $m(\vartheta,t), \dot{m}(\vartheta,t)$, correspondingly, when $\varepsilon \to 0$. It is easy to see that

$$m_0(\vartheta,t) = y_0 \exp\left[-\int_0^t a(s)ds \right]$$
$$\dot{m}(\vartheta,t) = \int_0^t \exp\left[-\int_s^t q(\vartheta,v)dv \right] h(\vartheta,s)ds \tag{32}$$

where

$$h(\vartheta,t) = \gamma_*(\vartheta,t)f(\vartheta t) \tag{33}$$

It is shown below (Lemma 1) that the corresponding family of measurers is locally asymptotically normal (LAN), and, therefore, we can define the lower bound (Hajek-Le Cam) for the mean square risk of any estimator $\overline{\vartheta}_\varepsilon$:

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \sup_{|\vartheta - \vartheta_0| < \delta} \varepsilon^{-2} \mathbf{E}_\vartheta |\overline{\vartheta}_\varepsilon - \vartheta| \geq I^{-1}(\vartheta_0) \tag{34}$$

We call an estimator ϑ_ε^* the asymptotically efficient, if for all $\vartheta_0 \in \Theta$ we have the equality in Equation (34).

Theorem 1. Referring to the compact $K \in \Theta$, the MLE and the BE are uniformly consistent and asymptotically normal:

$$(\hat{\vartheta}_\varepsilon - \vartheta_0)/\varepsilon \Rightarrow N(0, I_0^{-1}(\vartheta_0))$$
$$(\tilde{\vartheta}_\varepsilon - \vartheta_0)/\varepsilon \Rightarrow N(0, I_0^{-1}(\vartheta_0)) \tag{35}$$

and we have the convergence of all the polynomial moments $(p > 0)$:

$$\varepsilon^{-p} I_0^{p/2}(\vartheta_0)\mathbf{E}_{\vartheta_0}|\hat{\vartheta}_\varepsilon - \vartheta_0|^p \to \mathbf{E}|\zeta|^p,$$
$$\varepsilon^{-p} I_0^{p/2}(\vartheta_0)\mathbf{E}_{\vartheta_0}|\tilde{\vartheta}_\varepsilon - \vartheta_0|^p \to \mathbf{E}|\zeta|^p, \zeta \sim N(0,1) \tag{36}$$

and thus both estimators are asymptotically efficient.

Proof. The proof of this theorem follows from the general results of the Theorems 3.2.1 and 3.2.2 in [13]. The conditions for these theorems are fulfilled, if we prove the following three lemmas.

Lemma 1. The normalized likelihood ratio

$$Z_\varepsilon(u) = V(\vartheta_0 + \varepsilon u, X^T)$$
$$u \in U_\varepsilon = ((\alpha - \vartheta_0)/\varepsilon, (\beta - \vartheta_0)/\varepsilon) \tag{37}$$

allows the representation

$$Z_\varepsilon(u) = \exp\left[u\Delta_\varepsilon(\vartheta_0, X^T) - u^2 I_0(\vartheta_0)/2 + r_\varepsilon \right] \tag{38}$$

where $r_\varepsilon \to 0$ and

$$\Delta_\varepsilon(\vartheta_0, X^T) = \int_0^T \left[f(\vartheta t)\dot{m}_0(\vartheta,t) + tf'(\vartheta t)m_0(\vartheta,t) \right] \cdot$$
$$\cdot dW \sim N(0, I_0(\vartheta_0)) \tag{39}$$

Proof. Let us put that $M(\vartheta,t) = f(\vartheta t)m(\vartheta,t)$ and hence

$$\dot{M}_0(\vartheta,t) = f(\vartheta t)\dot{m}(\vartheta,t) + tf'(\vartheta t)m_0(\vartheta,t) \tag{40}$$

As it is easy to see, we have: $M(\vartheta_0 + \varepsilon u, t) - M(\vartheta_0, t) = \varepsilon u \dot{M}_0(\vartheta_0, t)(1 + o(1))$. Therefore,

$$\ln Z_\varepsilon(u) = \frac{1}{\varepsilon} \int_0^T [M(\vartheta_0 + \varepsilon u, t) - M(\vartheta_0, t)]dW_t -$$
$$-\frac{1}{2\varepsilon^2} \int_0^T [M(\vartheta_0 + \varepsilon u, t) - M(\vartheta_0, t)]^2 dt = \tag{41}$$
$$= u\Delta_\varepsilon(\vartheta_0, X^T) - u^2 I_0(\vartheta_0)/2 + r$$

where $r_\varepsilon \to 0$.

Lemma 2. There exists a constant $C > 0$, such as this

$$\mathbf{E}_{\vartheta_0}|Z_\varepsilon^{1/2}(u_2) - Z_\varepsilon^{1/2}(u_1)|^2 \leq C|u_2 - u_1|^2 \tag{42}$$

Proof. Let us put

$$Q_{t,\varepsilon} = \left[\frac{V(\vartheta_0 + \varepsilon u_2, X^t)}{V(\vartheta_0 + \varepsilon u_1, X^t)} \right]^{1/2},$$
$$\delta_t = M(\vartheta_0 + \varepsilon u_2, t) - M(\vartheta_0 + \varepsilon u_1, t) \tag{43}$$

Then, by applying the Ito formula, we come to the following relation

$$dQ_{t,\varepsilon} = -\frac{1}{8\varepsilon^2} Q_{t,\varepsilon}\delta_t^2 dt + \frac{1}{2\varepsilon} Q_{t,\varepsilon}\delta_t dW_T, \ Q_{0,\varepsilon} = 1 \tag{44}$$

and we can write

$$\mathbf{E}_{\vartheta_0}|Z_\varepsilon^{1/2}(u_2) - Z_\varepsilon^{1/2}(u_1)|^2 = \mathbf{E}_{\vartheta_1}(Q_{T,\varepsilon} - 1)^2 =$$
$$= 2 - 2\mathbf{E}_{\vartheta_1} Q_{T,\varepsilon} = \frac{1}{4\varepsilon^2} \int_0^T \mathbf{E}_{\vartheta_1} Q_{t,\varepsilon}\delta_1^2 dt \leq \tag{45}$$
$$\leq \frac{1}{8\varepsilon^2} \int_0^T \mathbf{E}_{\vartheta_1}\delta_1^2 dt + \frac{1}{8\varepsilon^2} \int_0^T \mathbf{E}_{\vartheta_2}\delta_1^2 dt \leq C|u_2 - u_1|^2$$

Here we have changed the measures: $\mathbf{E}_{\vartheta_1} Q_{t,\varepsilon}\delta_1^2 = \mathbf{E}_{\vartheta_2}\delta_1^2$, where $\vartheta_i = \vartheta_0 + \varepsilon u_i, i = 1, 2$, and the relations used are: $\mathbf{E}_{\vartheta_1} Q_{T,\varepsilon}^2 = 1$ and $2ab \leq a^2 + b^2$.

Lemma 3. There exists a constant $\kappa > 0$, such as this

$$\mathbf{E}_{\vartheta_1} Z_\varepsilon^{1/2}(u) \leq \exp(-\kappa u^2) \tag{46}$$

Proof. The proof of this lemma is omitted because it is cumbersome but it follows the same steps as the proof of the Lemma 2.4 in [10].

The properties of the normalized likelihood ratio $Z_\varepsilon(u)$ established in these three lemmas allows us to cite the Theorem 1.3.1 and 1.3.2 from [13] and, therefore, obtain the properties of MLE and BE, announced in the Theorem 1. Particularly, the convergence of the random process $Z_\varepsilon(u) \Rightarrow Z(u)$, where

$$Z(u) = \exp\left[u\Delta - u^2 I_0(\vartheta_0)/2 \right], \Delta \sim N(0, I_0(\vartheta_0)) \qquad (47)$$

allows one to obtain the asymptotic normality of the MLE in the following way:

$$
\begin{aligned}
P_{\vartheta_0}\left(\frac{\hat{\vartheta}_\varepsilon - \vartheta_0}{\varepsilon} < x \right) &= P_{\vartheta_0}(\hat{\vartheta}_\varepsilon < \vartheta_0 + \varepsilon x) = \\
&= P_{\vartheta_0}\left[\sup_{\vartheta < \vartheta_0 + \varepsilon x} V(\vartheta, X^T) > \sup_{\vartheta \geq \vartheta_0 + \varepsilon x} V(\vartheta, X^T) \right] = \\
&= P_{\vartheta_0}\left[\sup_{\vartheta < \vartheta_0 + \varepsilon x} \frac{V(\vartheta, X^T)}{V(\vartheta_0, X^T)} > \sup_{\vartheta \geq \vartheta_0 + \varepsilon x} \frac{V(\vartheta, X^T)}{V(\vartheta_0, X^T)} \right] = \\
&= P_{\vartheta_0}\left[\sup_{u < x, u \in U_\varepsilon} Z_\varepsilon(u) > \sup_{u \geq x, u \in U_\varepsilon} Z_\varepsilon(u) \right] \rightarrow \\
&\rightarrow P_{\vartheta_0}\left[\sup_{u < x} Z(u) > \sup_{u \geq x} Z(u) \right] = P_{\vartheta_0}(\hat{u} < x)
\end{aligned}
\qquad (48)
$$

Here we have changed the variable $\vartheta = \vartheta_0 + \varepsilon u$. It is easy to see that

$$\hat{u} = I_0^{-1}(\vartheta_0)\Delta \sim N(0, I_0^{-1}(\vartheta_0)) \qquad (49)$$

Thus, the asymptotic normality of the MLE is obtained. For the BE, the corresponding limits provide us the relation

$$\frac{\tilde{\vartheta}_\varepsilon - \vartheta_0}{\varepsilon} \Rightarrow \frac{\int_{-\infty}^{\infty} u Z(u)\, du}{\int_{-\infty}^{\infty} Z(u)\, du} = \frac{\Delta}{I_0(\vartheta_0)} \sim N(0, I_0^{-1}(\vartheta_0)) \qquad (50)$$

For the details see [13].

4. Discussion

It is of interest to see what would happen, if the rates $\varepsilon \to 0$, $\psi(\varepsilon) \to 0$ are different. Let us suppose that $\phi_\varepsilon = \varepsilon/\psi(\varepsilon) \to 0$. Then, this case would correspond to the situation when the noise in the state equation is essentially bigger than in the observation one. We are specifically concerned with the rate of the MLE convergence to the true value.

Let us consider the Riccati equation

$$\left[\frac{\partial \gamma_*(\vartheta, t)}{\partial t} + 2a(t)\gamma_*(\vartheta, t) \right]\phi_\varepsilon = \\ = -\gamma_*^2(\vartheta, t)f^2(\vartheta t) + b^2(t), \qquad \gamma_*(\vartheta, t) = 0 \qquad (51)$$

where we have $\gamma_*(\vartheta, t) = \varepsilon^{-1}\psi^{-1}(\varepsilon)\gamma(\vartheta, t)$. Our supposition is that the functions $a(t)$, $b(t)$ and $f(t)$ are strictly positive and bounded. It can be shown that, for any $t_0 > 0$, we have uniform convergence over $t \in [t_0, T]$

$$\gamma_*^2(\vartheta, t) = f^{-2}(\vartheta t)b^2(t)[1 + o(1)] \qquad (52)$$

This convergence allows us to study the asymptotics of $\dot{m}(\vartheta, t)$ and $\gamma_*(\vartheta, t)$, and, therefore, to describe the mean square error $\mathbf{E}_\vartheta(\hat{\vartheta}_\varepsilon - \vartheta)^2$. For example, if $\psi(\varepsilon) = c > 0$, then $\mathbf{E}_\vartheta(\hat{\vartheta}_\varepsilon - \vartheta)^2 = \varepsilon I_0^{-1}(\vartheta)[1 + o(1)]$, with a different Fisher information $I_0(\vartheta)$ [9]. Particularly, we show that $\varepsilon^{1/2}(\hat{\vartheta}_\varepsilon - \vartheta_0) \Rightarrow N(0, I_0^{-1}(\vartheta_0))$. The general case when $\psi(\varepsilon) \to 0$ and $\phi_\varepsilon \to 0$ is to be treated later.

5. Conclusion

Thus, maximum likelihood and Bayesian estimators of the frequency of the regular periodic signal in the presence of a multiplicative interference described by the Gaussian Markov random process and the additive distortions, like the white Gaussian noise, are asymptotically normal, consistent, effective, and have the rate of convergence inversely proportional to the voltage signal-to-noise ratio. The limiting variance of the estimators is determined by the Fisher information for which the expression in explicit form can be found analytically.

The domain of applicability of the obtained theoretical results (their satisfactory concordance with the corresponding experimental data) can be evaluated on the basis of the studies presented in [1].

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