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CLASSICAL SOLUTION AND EDGE EFFECT IN THE PROBLEM OF STABILITY OF AN AXIALLY COMPRESSED CYLINDRICAL SHELL

The classical solution for critical stresses in the problem of stability of a circular longitudinally compressed cylindrical shell consists of two terms, reflecting the ability of the shell to resist buckling due to bending and membrane deformations. However, with usual boundary conditions the classical solution appears only with the absence of the Poisson expansion of a shell. With a non-zero Poisson's ratio, an axisymmetric edge effect presents. It reduces the critical load and causes the initial arrangement of its own forms to change as the load increases.

Keywords: cylindrical shell, spherical shell, edge effect, loss of stability, competition of eigen forms

1 Introduction

It is known that the solution of the linear stability problem of an axially compressed circular elastic cylindrical shell without initial imperfections gives the classical value of the critical stresses

$$\sigma_{cr,cl} = E\delta/R\sqrt{3(1-\nu^2)}. \quad (1)$$

Buckling of the shell occurs either in an axisymmetric form or in a form that is a linear combination of axisymmetric ($n = 0$) and skew-symmetric ($n = 1$) eigen forms. Here n is the number of total circumferential waves in critical equilibrium.

Equation (1) was obtained by Lorenz (1908, non-axisymmetric buckling [1]) and Timoshenko (1910, symmetrical buckling [2]). However, in subsequent years it was established that if the shell material has a nonzero Poisson's ratio ($\nu \neq 0$), then in many cases of boundary conditions the classical critical load is not achieved even in calculations. The reason is a development of a non-linear axisymmetric edge effect, caused by the constraint with the boundary conditions of free Poisson expansion. In the book [3] Panovko wrote that the axisymmetric boundary effect makes the linear eigen value problem inhomogeneous, and to obtain the solution of the stability problem one should perform a geometrically nonlinear calculation. This remark is true in relation to axisymmetric eigen forms. But with respect to non-axisymmetric forms of loss of stability (forms of "wave formation"), the eigen value problem remains homogeneous. These forms of stability loss of an axially compressed cylindrical shell are unstable (watershed), since they separate the stable subcritical equilibrium from another stable but strongly deformed distant equilibrium, meaning the destruction of the shell [4-5].

Another important point is a significant decrease of the critical load with certain boundary conditions (for example, under the condition of a free edge and some others).

In general, this is due to the fact that the total resistance of the shell to buckling is created from the bending energy and the energy of the circumferential membrane deformations.

In this paper, the above-mentioned expansion of the critical load expression is considered into two terms, the solution is being analyzed, as well as results of the study of the influence of the edge effect by Almros [6-7]. It is shown that the edge effect leads to disappearance of the axisymmetric form of loss of stability. Some features of development of the edge effect and its role in the wave formation at the time of loss of stability were also studied. In addition, some variants of boundary conditions of the shell were considered that lead to reduction of the critical load by half in comparison with the classical one. In the paper, the periodic buckling modes of an axially compressed cylindrical shell are being examined. Matters of localized buckling modes are being considered in other works of the authors [4, 8].

2 On the classical critical load

The bifurcation problem of the stability of an axially compressed circular cylindrical shell is known [9] to be described by a linear partial differential equation with respect to the deflection $W(x, y)$ for zero boundary conditions

$$\frac{D}{\delta} \nabla \nabla \nabla \nabla W + \frac{E}{R^2} \frac{\partial^2 W}{\partial x^4} + \sigma \nabla \nabla \left(\frac{\partial^2 W}{\partial x^4} \right) = 0. \quad (2)$$

For the case of pinned edge

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$$W = \frac{\partial^2 W}{\partial x^4} = \sigma_x = V = 0 \quad \text{for } x = 0.1, \quad (3) \quad \Psi = (1 + \xi^2)^2 \eta / \xi^2. \quad (11)$$

where:

V - transverse displacement,

∇ - Laplace operator,

D - cylindrical rigidity,

σ - axial compressive stress.

As a solution (non-axisymmetric form of loss of stability), one can take

$$W = A_{\min} \sin \frac{\pi m x}{l} \sin \frac{n y}{R}, \quad (4)$$

where n and m are integers that determine the number of circumferential waves and axial half-waves. Substitution of the expression for the deflection in the form of a product of sinusoids makes it possible to express the critical stress in the following form:

$$\sigma_{cr} = \frac{E\partial}{R} \left[\frac{\partial^2 R}{12(1-v^2)\pi^2 m^2} \left(\frac{\pi^2 m^2}{l^2} + \frac{n^2}{R^2} \right) + \frac{\pi^2 m^2}{\partial^2 R \left(\frac{\pi^2 m^2}{l^2} + \frac{n^2}{R^2} \right)^2} \right] \quad (5)$$

If the dimensionless quantities are introduced

$$\eta = n^2 \delta / R; \quad \xi = b/a = \pi R m / n l, \quad (6)$$

where a and b are the dimensions of the rectangular wave formation cells along the generator and the cylinder guide. If one denotes

$$\frac{\pi^2 m^2 R^4}{\delta l^2 R n^4} = \frac{\pi^2 m^2 R^2}{l^2 n^2} \cdot \frac{R}{\delta n^2} = \xi^2 / \eta, \quad (7)$$

then one can write

$$\sigma_{cr} = \frac{E\delta}{R} \overline{\sigma}_{cr} = \frac{E\delta}{R} \left[\frac{(1 + \xi^2)\eta^2}{12(1-v^2)\xi^2} + \frac{\xi}{(1 + \xi^2)\eta} \right], \quad (8)$$

or

$$\sigma_{cr} = \frac{E\delta}{R} [\overline{\sigma}_b + \overline{\sigma}_m] = \frac{E\delta}{R} \overline{\sigma}_{cr}. \quad (9)$$

Here the first term in parentheses determines the energy contribution of the flexural deformations of the shell to the total buckling resistance; the second term is the contribution of the energy of circumferential membrane deformations. The expression in square brackets consists of two actually reciprocal quantities. Therefore, the smallest value of the critical stress is achieved when these terms are equal. Then, each of them is equal to half of the minimum critical load

$$\sigma_{cr}^{\min} \rightarrow \sigma_{cr}^b = \sigma_{cr}^m = \frac{1}{2} \sigma_{cr}^{\min}. \quad (10)$$

To determine σ_{cr}^{\min} it is convenient to introduce the notation

Then, according to what has been said, follows:

$$\overline{\sigma}_{cr}^{\min} = \left[\frac{\Psi}{12(1-v^2)} + \frac{1}{\Psi} \right] \rightarrow \Psi^2 = 12(1-v^2); \quad (12)$$

$$\Psi = 2\sqrt{3(1-v^2)}.$$

Finally, one obtains an expression for the classical critical stress

$$\sigma_{cr}^{\min} = \frac{E\delta}{R} \left(\frac{1}{2\sqrt{3(1-v^2)}} + \frac{1}{2\sqrt{3(1-v^2)}} \right) = \frac{E\delta}{2\sqrt{3(1-v^2)}}. \quad (13)$$

The number of circumferential waves n depends on the ratio of the sides of the rectangular wave formation cells $\xi = a/b$

$$n = \frac{\xi}{1 + \xi^2} \sqrt{\frac{R^2}{\delta^2} 12(1-v^2)}. \quad (14)$$

For $\xi = 1$ (square cells) the well-known equation is obtained $n \cong 0.909 \sqrt{\frac{R}{\delta}}$.

The energy justification of Equation (9) will be shown in deriving a critical force ($P = \sigma_{cr} \delta$), as Timoshenko did [10]. It used the sinusoidal form of deflections ($w = A \sin \frac{\pi m x}{R}$) as an axisymmetric form of loss of stability.

The condition for achieving the critical equilibrium can be represented as the vanishing of the second variation of the total potential energy

$$\delta^2 E = I_\sigma + I_b + I_m = 0, \quad (15)$$

where I_σ is the work of the external load,

I_b and I_m is the potential energy of bending and membrane deformations,

$$I_\sigma = -2\pi R P \int_0^1 \frac{1}{2} \left(\frac{dW}{dx} \right)^2 dx = -P A^2 \pi^3 m^2 R / 21, \quad (16)$$

$$I_b = 2\pi R D \int_0^1 \frac{1}{2} \left(\frac{d^2 W}{dx^2} \right)^2 dx = A^2 D \pi^4 m^4 R / 21^3, \quad (17)$$

$$I_m = \frac{2\pi R E \delta}{2(1-v^2)} \int_0^1 \left[\left(\frac{du}{dx} - \frac{w}{R} \right)^2 + 2(1-v) \frac{w du}{R dx} \right] dx = A^2 \pi E \delta / R. \quad (18)$$

The critical load is obtained in the form of two components in accordance with the representation of the energy of elastic deformation as the sum of the energy of the bending deformation and the energy of the membrane deformation.

$$P_{cr} = \frac{E\delta^3 2m^2 \pi^2}{24(1-v^2)l^2} + \frac{E\delta l^2}{\pi^2 m^2 R^2} = \frac{E\delta^2}{R} \overline{P}_{cr}, \quad (19)$$

or

$$P_{cr} = \frac{E\delta^2}{R} \left(\frac{\delta R \pi^2 m^2}{12(1-v^2)l^2} + \frac{l^2}{\delta R \pi^2 m^2} \right) = \frac{E\delta^2}{R} (\bar{P}_b + \bar{P}_m). \quad (20)$$

In the above derivation of the equation of the classical critical stress, it corresponds to the equality of the terms in parentheses

$$\min P_{cr} \Rightarrow \bar{P}_b = \bar{P}_m = \frac{1}{2} \bar{P}_{cr-cl}. \quad (21)$$

If one denotes

$$1/\psi = \delta R \pi^2 m^2 / l^2; \quad 1/\psi = \bar{P}_m, \quad (22)$$

then

$$\bar{P}_m = \bar{P}_b = \frac{1}{2\sqrt{3(1-v^2)}} = \frac{1}{2} \bar{P}_{cr-cl}. \quad (23)$$

Finally, one gets

$$P_{cr} = \frac{E\delta^2}{R} \left(\frac{1}{2\sqrt{3(1-v^2)}} + \frac{1}{2\sqrt{3(1-v^2)}} \right) = \frac{E\delta^2}{R\sqrt{3(1-v^2)}}. \quad (24)$$

The number of axial half-waves m is defined as follows:

$$m = \sqrt{2\sqrt{3(1-v^2)}} / \pi \sqrt{R\delta}. \quad (25)$$

It should be noted that the decomposition of the critical force into a bending and equal membrane component is valid only for the classical solution of the linearized stability problem considered above. In the case of the development of the edge effect, the ratio between \bar{P}_b and \bar{P}_m can change. At the same time, under certain boundary conditions, it can turn out that \bar{P}_m is much smaller \bar{P}_b , and then the critical force \bar{P}_{cr} will be much lower than the classical value.

3 Influence of the edge effect

Studies of the edge effect on the axially compressed cylindrical shells were carried out by Geckeler [11], Fisher [12], Ohira [13], Almros [6], Hoff and Reifeld [14]. Especially important are the results obtained by Almros. He presented the general solution for deflection W and stress function f at the moment of loss of stability in the form of two terms

$$W = W_0 + W_1; \quad f = f_0 + f_1. \quad (26)$$

The first terms are solutions of the differential equation of the boundary effect (for details see [6])

$$Dw_0xxxx + Pw_0xx + E\delta w_0/R^2 - \frac{vP}{R} = 0. \quad (27)$$

The second summand are zero eigen modes ($W_1 = \xi W_{c\varphi}^0$), which are found by numerically solving the homogeneous equilibrium and deformation compatibility equations, after substituting in them the solutions for the edge effect in them W_0 and f_0

$$D\nabla^4 W_1 + f_{1,xx/R} + PW_{1,xx} - W_{0,xx}f_{1,yy} + \left(vP - \frac{E\delta W_0}{R}\right)W_{1,yy} = 0, \quad (28)$$

$$\frac{1}{E\delta} \nabla^4 f_1 - W_{1,xx/R} + W_{0,xx}W_{1,yy} = 0. \quad (29)$$

Here, the subscripts xx and yy after the comma mean differentiation with respect to the corresponding coordinate. Using

$$f_1 = F(x) \sin \frac{ny}{R}; \quad W_1 = W(x) \sin \frac{ny}{R}. \quad (30)$$

Almros moved to a system of ordinary equations with variable coefficients with respect to $F(x)$ and $W(x)$

$$D \left[W'''' - 2 \left(\frac{n}{R} \right)^2 W'' + \left(\frac{n}{R} \right)^4 W \right] + \frac{F}{R} + PW'' + \left(\frac{n}{R} \right)^2 W_0'' F - \left(\frac{n}{R} \right)^2 \cdot \left(vP - \frac{E\delta W_0}{R} \right) W = 0, \quad (31)$$

$$\frac{1}{E\delta} \left[F'''' - 2 \left(\frac{n}{R} \right)^2 F'' + \left(\frac{n}{R} \right)^4 F \right] - \frac{W''}{R} - \left(\frac{n}{R} \right)^2 W_0'' W = 0. \quad (32)$$

Boundary conditions were expressed through W and F in the usual way in the case when they include W, f, P, N_{xy} . The conditions for displacements were written as follows:

$$U = 0 \rightarrow F'' + vF \left(\frac{n}{R} \right)^2 = 0, \quad (33)$$

$$V = 0 \rightarrow F'' - (2+v)F \left(\frac{n}{R} \right)^2 - E\delta W' = 0. \quad (34)$$

A classical solution is obtained if one assumes that

$$W_0 = vPR/E\delta. \quad (35)$$

The solution of equations and determination of the critical load were carried out numerically by the method of finite differences (MFD).

The homogeneous problem of determining the bifurcation load corresponding to a non-axisymmetric form of loss of stability was solved from the condition that the determinant of the corresponding matrix be zero.

Studies of the convergence of MFD showed that to obtain two valid signs of the critical load, it is sufficient to have 50-100 sampling points. Refinement of the critical load (3-4 valid marks) required a substantial grinding of the grid (up to 400-600 points) and, correspondingly, an increase in the labor intensity of problem solving.

In problems with pinned boundary condition, convergence from below was observed, whereas in the

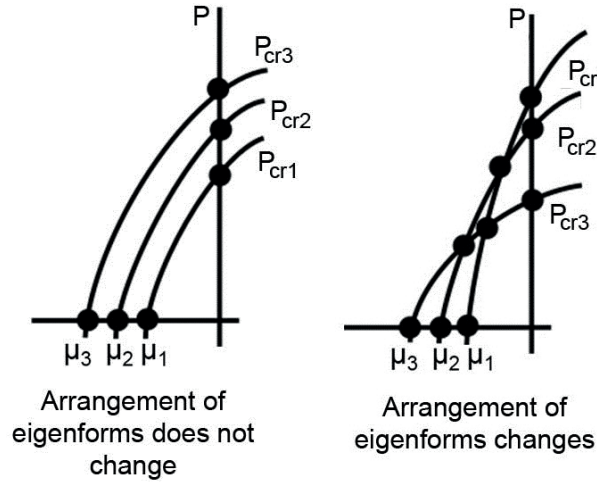


Figure 1 Competition of eigen modes

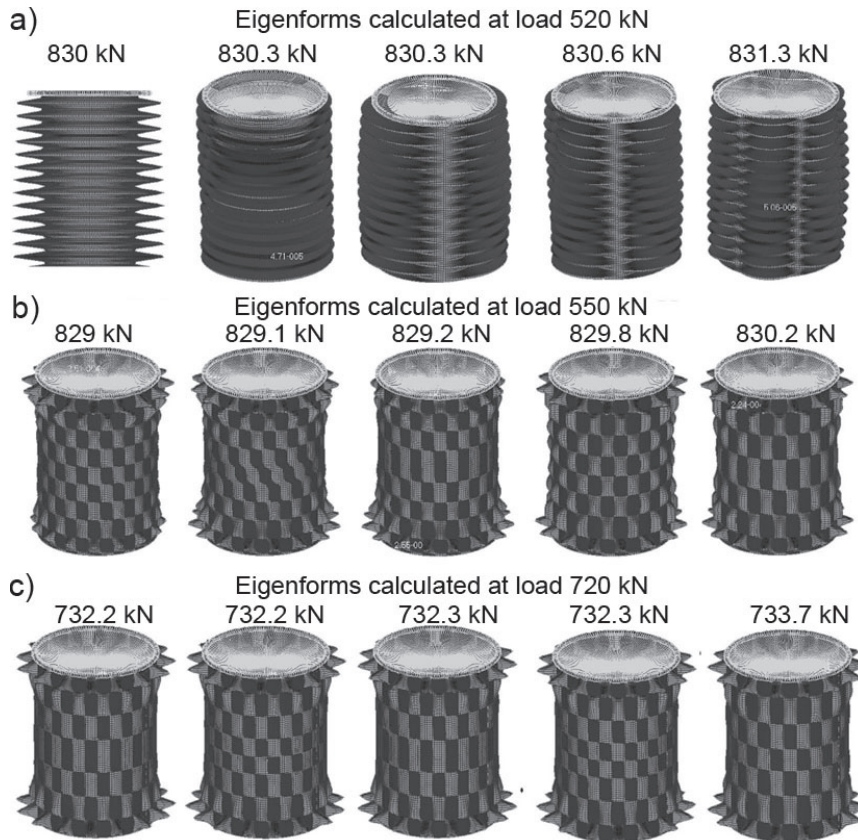


Figure 2 Eigen forms, calculated at loads of 520, 550 and 720 kN ($\nu = 0.3$)

case of fixed edges of the shell, the convergence of the approximate values of the critical load was from above.

The results of this investigation with respect to $\eta = \frac{\sigma_{cr}^{cr-eff}}{\sigma_{cr-cl}}$ with eight different combinations of boundary conditions are given in the article's table [6]. Note that Almros used not all combinations of R/δ and R/δ .

As follows from results in that table, influence of those shell parameters on the critical load is much less than the influence of the boundary conditions. Thus, for standard hinge support conditions

$$S_1(W|_r = W'|_r = N_x|_r = V|_r = 0) \text{ at } 4R > \frac{1}{2}, \quad (36)$$

$$\eta = \frac{\sigma_{cr}^{cr-eff}}{\sigma_{cr-cl}} \approx 0.8 - 0.87, \quad (37)$$

under the boundary condition

$$S_2(w|_r = w'|_r = u|_r = V|_r = 0) \eta \approx 0.84, \quad (38)$$

$$\text{for } S_3(w|_r = w'|_r = N_x|_r = N_{xy}|_r = 0) \text{ value } \eta \text{ smaller } (\eta \approx 0.5 \div 0.51). \quad (39)$$

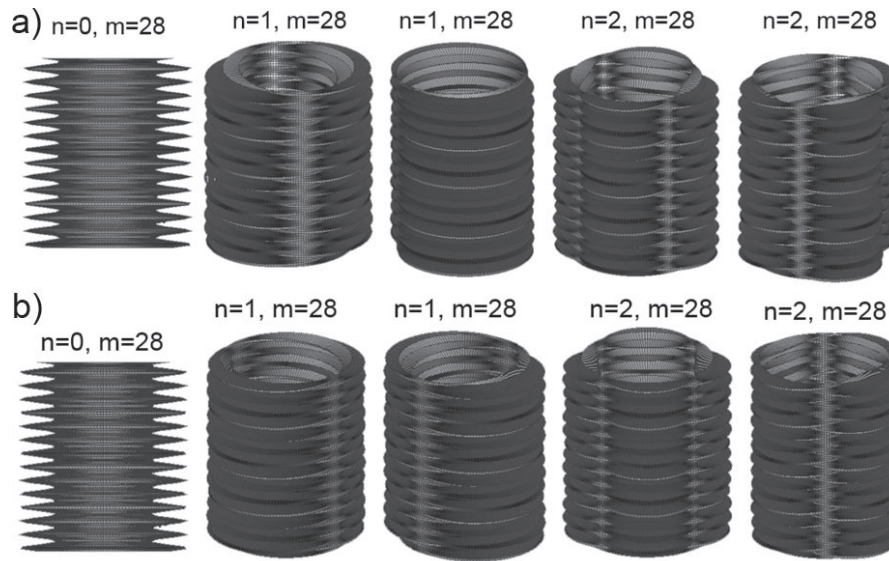


Figure 3 The eigen forms of a shell with zero Poisson's ratio

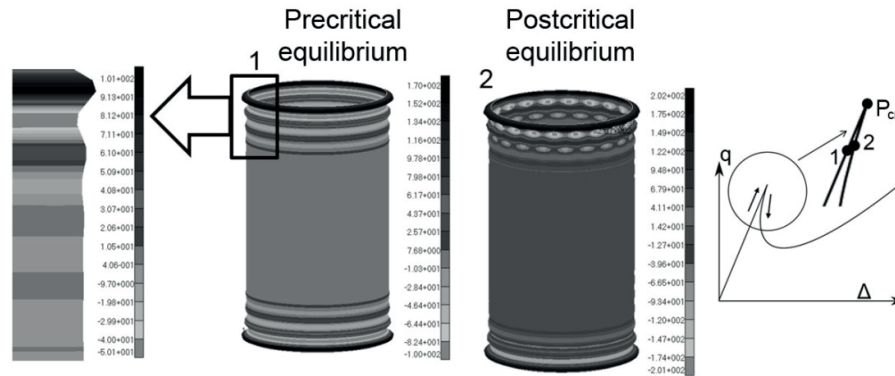


Figure 4 Distribution of stresses in the circumferential direction in pre critical and post critical equilibrium

Practically the same values η ($\eta \cong 0.5 \div 0.51$) were obtained for the case

$$S_4(w|_r = w'|_r = N_x|_r = N_{xy}|_r = 0). \quad (40)$$

It was noted that in the case of S_3 and S_4 , the reduction in the critical load is the same as without taking into account the edge effect. These last two results (S_3 and S_4) indicate an important inhibiting role of circumferential membrane stresses. In the absence of these stresses, the critical load is almost halved compared to the classical one.

In the case of a standard fixed edge (the condition $C_3(w|_r = W'|_r = N_x|_r = N_{xy}|_r = 0)$), as in the case of the condition $C_1(w|_r = W'|_r = u|_r = V|_r = 0)$, reduction of the load somewhat less $\eta = 0.91 \div 0.93$.

A slightly smaller value η for the case $C_2(C_2(w|_r = W'|_r = N_x|_r = v|_r = 0))$.

Here $\eta \cong 0.86 \div 0.91$. Approximately the same values η are for the combination $C_4(C_4(W|_r = W'|_r = N_x|_r = N_{xy}|_r = 0))$, $\eta \cong 0.86 \div 0.91$.

It was found that the form of a non-axisymmetric loss of stability has 8-9 circumferential waves for all

the cases except for S_3 and S_4 , for which the number of circumferential waves decreased to two ($n_{cr} = 2$).

The finite element calculations of the authors (NASTRAN, MARC) showed a critical pressure $P_{cr} = 713$ kN for the hinged shell of Croll [15] ($R/\delta = 300$, $R = 300$ mm, $L/R = 2.88$, $E = 2.1 \cdot 10^4$ kg/mm², $\nu = 0.3$). For a close-sized shell ($R/\delta = 100$, $L/R = 3.2$) with the same pinned edge Almros obtained $P \sim 693$ kN. The classical critical load for this shell is:

$$P_{cr-cl} = 1.21\pi E \delta^2 = 798.28 \text{ kN}. \quad (41)$$

The Almros's result is $\eta = 0.868$. Here the result is slightly larger ($\eta = 0.893$). The number of circumferential waves is $n = 15$, and not 9 as in Almros, which is explained by a much larger value of $R/\delta = 250 > 100$. It is noted that the development of a nonlinear axisymmetric edge effect due to the constraint of the Poisson expansion of the shell's edge leads to a competition of eigen modes as the compressive load increases. This means that the curves of changes in the corresponding eigenvalues, with respect to the load of the derivative of the nonlinear operator, intersect each other.

Thus, the mutual arrangement of the eigen forms of loss of stability changes (Figure 1).

Calculations in this paper for a number of hinged and fixed shells showed that the axisymmetric and skew-symmetric forms of loss of stability remain first up to a load of $(0.65-0.7) P_{cr}$. However, further, these forms “yield” the first place (corresponding to the least rigidity of the shell) to skew-symmetric forms. A particularity of these forms is the development of waves of the greatest amplitude near the edge of the shell. When from the edges, the wave amplitude decreases sharply [16]. This change in forms of loss of stability occurs with simultaneous more intensive development of the edge effect (as the compressive load approaches the critical value). When the load reaches the pre-critical value of the axisymmetric and skew-symmetric forms of loss of stability, there are no among the first ten eigen forms of the shell (Figure 2). At this point, the first two “zero” eigen forms W_1^0 and W_2^0 are the same. These are cyclically symmetrical double waveforms, rotated relative to each other by 6° . Therefore, the initial form of loss of stability is formed as a linear combination of these forms, which is summed with an axisymmetric pre-critical equilibrium

$$\bar{W}_{f,post/cr.} = \bar{W}_{precr.} + \xi_1 \bar{W}_1^0 + \xi_2 \bar{W}_2^0. \quad (42)$$

When can a classical solution be actually used? The answer is simple - when the edge effect will not develop (the standard conditions for hinging and fixing are meant). However, with such boundary conditions, one can avoid constraining Poisson expansion if the Poisson's ratio is zero ($\nu = 0$). In this case, the linearized classical solution of the stability problem can be used

$$\sigma_{cr} = \frac{E\delta}{\sqrt{3}R}. \quad (43)$$

In this case, the first eigen form of loss of stability is axisymmetric. But it can be a multiple.

Figure 3 shows eigen forms for the Kroll's shell at zero load (a) and for pre-critical equilibrium (b), ($P = 760 \text{ kN}$,

$P_{cr,p} = 761.8 \text{ kN}$). It can be seen that the initial order of the arrangement of the first 5 eigen forms is preserved up to the critical equilibrium. Consequently, among these eigen forms there is no competition in the absence of an edge effect. In this paper it is established that even a very small nonzero Poisson's ratio ($\nu = 0.05$) causes development of the edge effect.

It is noted that, due to the edge effect, a powerful belt of large tensile circumferential membrane stresses is created in the edge zone of the hinged shell. Such a belt (as a hoop) substantially increases the stability margin of a longitudinally compressed shell (Figure 4). Simultaneously with this belt a belt of significant compressive circumferential membrane stresses is formed. Here, the shell experiences a biaxial contraction - this is the most provocative zone from the point of view of the onset of wave formation. Indeed, upon transition to the initial post critical equilibrium, it is precisely along the zone of this compression belt that the first “wave formation elements” develop (Figure 4), which then expands to the entire shell surface.

4 Conclusion

Calculations performed on model shells showed that the development of a nonlinear edge effect leads to a change in the order of the eigen forms of loss of stability of the cylindrical shell. For model shells with a pivotally fixed edge, the axisymmetric form of buckling remained the first up to a load equal to $(0.65-0.7) P_{cr}$. Upon exceeding the indicated load values, the cyclically symmetric forms of buckling were becoming the first. The axisymmetric form of buckling remained the first until the critical load was reached on shell models where a nonlinear edge effect does not develop. This can be achieved with a zero Poisson's ratio, as well as when fixing the shell at its edges only with tangent supports (they prevent movement along the circumferential coordinate). However, the last fixing option (absence of radial support) leads to a significant reduction in the critical load.

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