MODELLING OF CRITICAL VELOCITIES OF THE CARDAN MECHANISM USING TRANSFER MATRIX METHOD

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Resume

The presented paper focuses to rotating components of mechanical constructions. The problem of the spatial combined bending-gyratory vibration and calculation of the Eigen frequencies is studied. The model of Cardan Mechanism is solved by the transfer matrix method. Transfer matrices were derived for shaft, concentrated mass and elastic bearing. The physical and mechanical properties of each part of the mechanism are hidden in these matrices. A procedure for calculating Eigen frequencies was proposed.

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1 Introduction

In mechanical constructions, the most endangered parts are rotating components, e.g. shafts [1]. Reliability of a shaft is endangered in particular in the two limit states. In the vicinity of resonance, there is an enormous increase in the amplitudes of the state variables and reaching of the yield strength of a material. These conditions often occur with the coupling shafts of Cardan mechanisms. The torque is transmitted here over long distances. Shafts are long and slender and are prone to transverse bending. The gearbox shafts are compact and operate at a sufficient distance from the resonant area. In that case, they are threatened by fatigue fractures; they need to be checked for safety to fatigue. A similar situation to gearboxes is with the gear pump shafts.

Mathematical models of the Cardan mechanism lead to solutions from the field of linear algebraic equations. In the case of bending oscillations, the motion equation of the basic element is a partial differential equation of the 4th order for the variables x and t [2]. An analytical solution for simpler cases can be used.

It seems appropriate, for this case, to use the transfer matrix method. This method does not increase the matrix size (matrix 4 x 4 for planar oscillation, 8 x 8 for spatial oscillation), resulting in the lower hardware requirements. The transfer matrix method uses a combination of analytical and numerical methods. The benefit of this calculation is a possibility to obtain deformations caused by external excitation and dynamic deformation and stress analysis. Using the transfer matrix method is relatively easy to get a solution to the whole system (the whole Cardan mechanism). Another advantage is that it can be combined with the method of the imaginary slice, which analytically solves the differential equations of motion for a smooth shaft (smooth continuum - a constant diameter), the transfer matrices for the shaft, matrices of concentrated mass and the elastic bearing, which are the basic structural elements of a dynamic model of shafts, are derived.

This paper is devoted to studying the problem spatial combined bending-gyratory vibration and calculation of the Eigen frequencies using the transfer matrix method.

2 Spatial combined bending-gyratory vibration

The element (see Figure 1) is an one-dimensional continuum with geometrical parameters as inner radius r_1, outer radius r_2 and length l. The physical parameters are E module of elasticity and density ρ. The whole system is rotating with angular velocity ω.

External forces, acting on the element, create a state of the combined bending-gyratory vibration. The continuum element is making general spatial motion which is composed

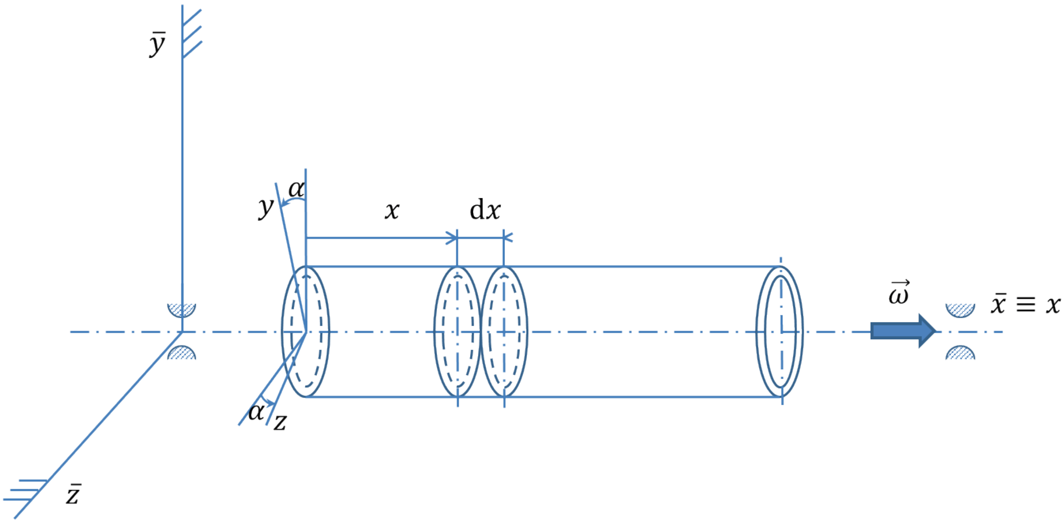

Figure 1 *The element of the continuum*

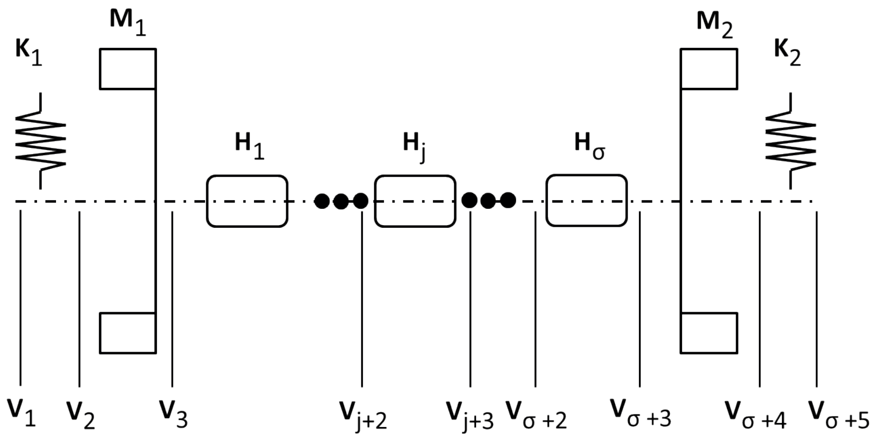

Figure 2 *Model of the whole Cardan mechanism*

of three simple movements - namely rotation, shift and spherical motion.

The following formula represents the equation of motion in the state of spatial combined bending-gyratory vibrations [2]

$$\frac{\partial^4 v}{\partial x^4} - \frac{\rho}{E}\left(\frac{\partial^4 v}{\partial x^2 \partial t^2} + \omega^2 \frac{\partial^2 v}{\partial x^2}\right) + \\ + \frac{4\rho}{E(r_2^2 + r_1^2)}\left(\frac{\partial^2 v}{\partial t^2} - \omega^2 v + 2i\omega \frac{\partial v}{\partial t}\right) = 0, \tag{1}$$

where

$$v(x,t) = y(x,t) + iz(x,t). \tag{2}$$

For more details see [2].

3 Derivation of the transfer matrix

The Transfer Matrix Method (TMM) is a combination of numerical and analytical methods and comes from the exact analytical solution (PDE of 4^{th} order) (see Equation 1).

It is necessary to derive the transfer matrices (matrices are denoted by bold letters, e.g. **H, M, K**) for each of the basic structural elements of the Cardan mechanism: shaft **H**, concentrated mass **M** and elastic bearing **K** (see Figure 2).

One needs to define the vector of state $\mathbf{V}_i$ on the edge cuts of each element based on amplitudes of state variables.

Vector of state (see Equation (2)):

$$\mathbf{V}(x,t) = \mathbf{V}(x)e^{i\omega t}, \mathbf{V}(x) = [\mathbf{Y}(x)|\mathbf{Z}(x)], \tag{3}$$

$$\mathbf{V}(x) = \begin{bmatrix} y(x) \\ y'(x) \\ -M_z(x) \\ -Q_y(x) \end{bmatrix}, \mathbf{Z}(x) = \begin{bmatrix} z(x) \\ z'(x) \\ -M_y(x) \\ -Q_z(x) \end{bmatrix}, \tag{4}$$

where $y(x)$ is the amplitude of deflection, $y'(x)$ is the slope of deflection, $M_z(x)$ is the amplitude of the bending moment and $Q_y(x)$ is the moving force.

Relationship between neighbouring state vectors is $\mathbf{V}_{i+1} = \mathbf{H}_j\mathbf{V}_i$, where $j = 1, ..., \sigma$, i, $i = j + 2$, $\mathbf{V}_2 = \mathbf{K}_1 \mathbf{V}_1$, $\mathbf{V}_3 = \mathbf{M}_1\mathbf{V}_2$..., $\mathbf{V}_{\sigma+5} = \mathbf{P} \cdot \mathbf{V}_1$.

The method uses knowledge of boundary conditions of the state vectors of the joint shaft. The frequency equation can then be determined from the coupling relations between the edge state vectors. The first and the last state vectors of the dynamic model are $\mathbf{V}_1$, resp. $\mathbf{V}_{\sigma+5}$, $\mathbf{V}_1[v_1(0), v_1'(0), 0, 0]^T$, $\mathbf{V}_{\sigma+5} = [0, 0, v_0(l_0), v_0'(l_0)]$.

States vectors $\mathbf{V}_1$ and $\mathbf{V}_{\sigma+5}$ are joint

$$\mathbf{V}_{\sigma+5} = \mathbf{P} \cdot \mathbf{V}_1 \tag{5}$$

The Transfer Matrix **P** is:

$$\mathbf{P} = \mathbf{K}_2 \cdot \mathbf{M}_2 \cdot \mathbf{H}_\sigma \ldots \mathbf{H}_j \ldots \mathbf{H}_1 \cdot \mathbf{M}_1 \cdot \mathbf{K}_1. \tag{6}$$

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_y & 0 \\ 0 & \mathbf{P}_z \end{bmatrix}, \mathbf{P}_y = \mathbf{P}_z = [p_{ij}]_1^4. \tag{7}$$

All the matrices in Equation (5) are of the 8×8 type. Equation (5) describes a system of eight equations of eight unknowns. The matrix **P** is blocked diagonal, so this system can be divided into two systems of 4 equations by 4 unknowns. Each system describes the movement of individual axes.

After entering the edge vectors, the transfer matrix and matrix multiplication are

$$\mathbf{A}_y \cdot \mathbf{B}_y = \mathbf{D}_y, \mathbf{A}_z \cdot \mathbf{B}_z = \mathbf{D}_z. \tag{8}$$

4 Solution to the problem

The following matrices, concerning Equations (5) to (7), are obtained.

$$\mathbf{A}_y = \mathbf{A}_z \begin{bmatrix} p_{11} & p_{12} & -1 & 0 \\ p_{21} & p_{22} & 0 & -1 \\ p_{31} & p_{32} & 0 & 0 \\ p_{41} & p_{42} & 0 & 0 \end{bmatrix}. \tag{9}$$

$$\mathbf{B}_y = \begin{bmatrix} y_1(0) \\ y_1'(0) \\ y_0(l_0) \\ y_0'(l_0) \end{bmatrix}, \mathbf{B}_z = \begin{bmatrix} z_1(0) \\ z_1'(0) \\ z_0(l_0) \\ z_0'(l_0) \end{bmatrix}. \tag{10}$$

$$\mathbf{D}_y = M_{1z} \cdot \begin{bmatrix} p_{13} \\ p_{23} \\ p_{34} - M_{2z}/M_{1z} \\ p_{43} \end{bmatrix}, \quad \mathbf{D}_z = M_{1y} \cdot \begin{bmatrix} p_{13} \\ p_{23} \\ p_{34} - M_{2y}/M_{1y} \\ p_{43} \end{bmatrix}. \tag{11}$$

The solution of the left edge of the joint shaft is:

$$y_1(0) = \frac{p_{42}(p_{33}M_{1z} - M_{2z}) - p_{32}p_{43}M_{1z}}{p_{31}p_{42} - p_{32}p_{41}}. \tag{12}$$

$$y_1'(0) = \frac{p_{31}p_{43}M_{1z} - p_{41}(p_{33}M_{1z} - M_{2z})}{p_{31}p_{42} - p_{32}p_{41}}. \tag{13}$$

$$z_1(0) = \frac{p_{42}(p_{33}M_{1y} - M_{2y}) - p_{32}p_{43}M_{1y}}{p_{31}p_{42} - p_{32}p_{41}}. \tag{14}$$

$$z_1'(0) = \frac{p_{31}p_{43}M_{1y} - p_{41}(p_{33}M_{1y} - M_{2y})}{p_{31}p_{42} - p_{32}p_{41}}. \tag{15}$$

5 The transfer matrix for a shaft

The transfer matrices for shafts parts have the following form

$$\mathbf{H}(x) = \begin{bmatrix} \mathbf{H}_y(x) & 0 \\ 0 & \mathbf{H}_z(x) \end{bmatrix}, \mathbf{H}_y = \mathbf{H}_z = , \quad = [\mathbf{H}_{11} | \mathbf{H}_{12} | \mathbf{H}_{13} | \mathbf{H}_{14}] \tag{16}$$

$$\mathbf{H}_{11} = \frac{1}{\beta_1^2 + \beta_2^2} \begin{bmatrix} \beta_2^2 \cosh \beta_1 l + \beta_1^2 \cos \beta_2 l \\ \beta_1 \beta_2 (\beta_2 \sinh \beta_1 l - \beta_1 \sin \beta_2 l) \\ EJ\beta_1^2\beta_2^2(\cosh \beta_1 l - \cos \beta_2 l) \\ EJ\beta_1^2\beta_2^2(\beta_1 \sinh \beta_1 l + \beta_2 \sin \beta_2 l) \end{bmatrix}, \tag{17}$$

$$\mathbf{H}_{12} = \frac{1}{\beta_1^2 + \beta_2^2} \begin{bmatrix} \beta_2^2/\beta_1 \sinh \beta_1 l + \beta_1^2/\beta_2 \sin \beta_2 l \\ \beta_2^2 \cosh \beta_1 l + \beta_1^2 \cos \beta_2 l \\ EJ\beta_1\beta_2(\beta_2 \sinh \beta_1 l - \beta_1 \sin \beta_2 l) \\ EJ\beta_1^2\beta_2^2(\cosh \beta_1 l + \cos \beta_2 l) \end{bmatrix}, \tag{18}$$

$$\mathbf{H}_{13} = \frac{1}{\beta_1^2 + \beta_2^2} \begin{bmatrix} 1/EJ(\cosh \beta_1 l - \cos \beta_2 l) \\ 1/EJ(\beta_1 \sinh \beta_1 l + \beta_2 \sin \beta_2 l) \\ \beta_1^2 \cosh \beta_1 l + \beta_2^2 \cos \beta_2 l \\ \beta_1^3 \sinh \beta_1 l + \beta_2^3 \sin \beta_2 l \end{bmatrix}, \tag{19}$$

$$\mathbf{H}_{14} = \frac{1}{\beta_1^2 + \beta_2^2} \begin{bmatrix} 1/EJ(1/\beta_1 \sinh \beta_1 l - 1/\beta_2 \sin \beta_2 l) \\ 1/EJ(\cosh \beta_1 l + \cos \beta_2 l) \\ \beta_1 \sinh \beta_1 l + \beta_2 l \\ \beta_1^2 \cosh \beta_1 l + \beta_2^2 \cos \beta_2 l \end{bmatrix}, \tag{20}$$

where:

$$J = \frac{\pi}{4}(r_2^4 - r_1^4), \tag{21}$$

$$\beta_1 = \left\{ -\frac{\rho}{2E}(\overline{\omega}^2 - \omega^2) + \left[\frac{\rho^2}{4E^2}(\overline{\omega}^2 - \omega^2)^2 + \frac{4\rho(\overline{\omega} - \omega)^2}{E(r_2^2 + r_1^2)} \right]^{\frac{1}{2}} \right\}^{\frac{1}{2}}. \tag{22}$$

$$\beta_2 = \left\{ \frac{\rho}{2E}(\overline{\omega}^2 - \omega^2) + \left[\frac{\rho^2}{4E^2}(\overline{\omega}^2 - \omega^2)^2 + \frac{4\rho(\overline{\omega} - \omega)^2}{E(r_2^2 + r_1^2)} \right]^{\frac{1}{2}} \right\}^{\frac{1}{2}}. \tag{23}$$

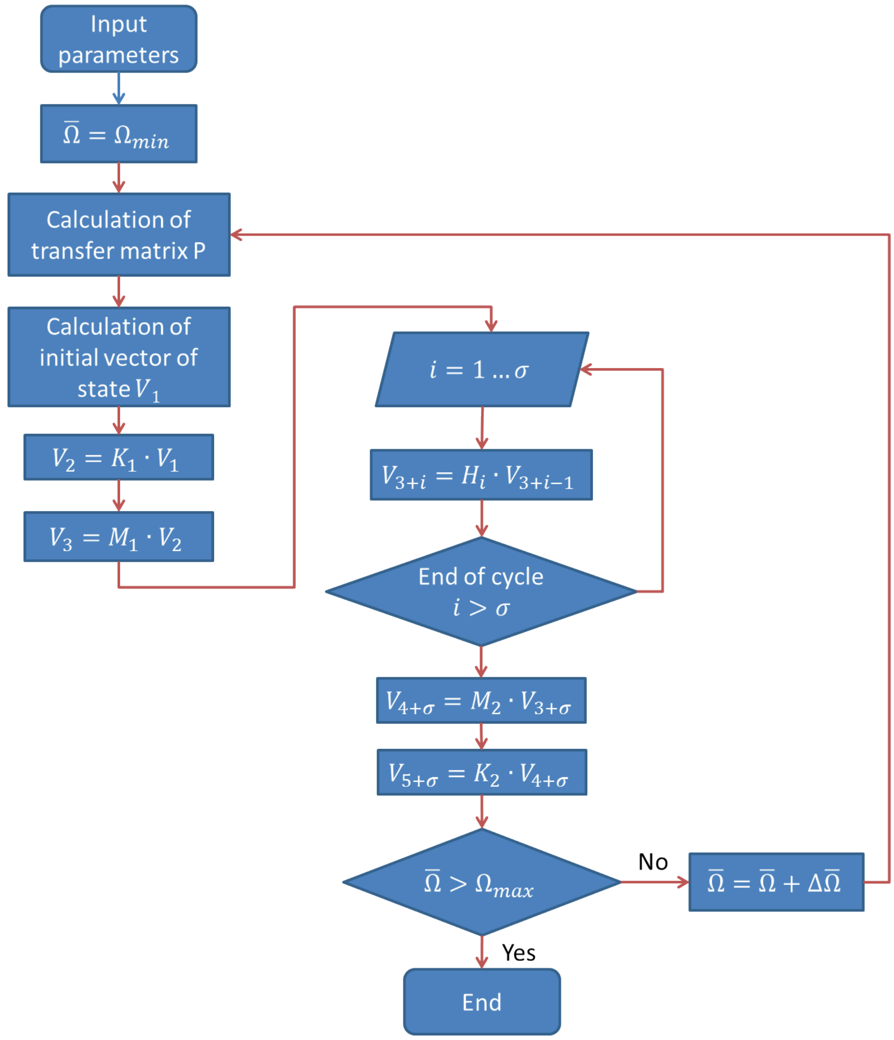

Figure 3 *Process of the Eigen frequencies calculations*

6 The transfer matrix for the concentrated mass

The matrices for concentrated masses are:

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_y & 0 \\ 0 & \mathbf{M}_z \end{bmatrix}. \quad (24)$$

$$\mathbf{M}_y = \mathbf{M}_z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -J_1\omega^2 + (J_0 - J_1)\omega^2 & 1 & 0 \\ m(\varpi + \omega^2) & 0 & 0 & 1 \end{bmatrix} \quad (25)$$

7 Transfer matrix for elastic bearing

Matrices for the elastic bearing can be expressed as:

$$\mathbf{K}(x) = \begin{bmatrix} \mathbf{K}_y & 0 \\ 0 & \mathbf{K}_z \end{bmatrix}. \quad (26)$$

$$\mathbf{K}_y = \mathbf{K}_z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -k & 0 & 0 & 1 \end{bmatrix}. \quad (27)$$

8 Eigen frequencies

Rewriting of Equation (8) leads to the following equation

$$\mathbf{F} \cdot \mathbf{V} = 0, \quad (28)$$

and

$$\mathbf{V} = [v(0), v'(0), v_0(l_0), v_0'(l_0)]^T. \quad (29)$$

One obtains set of equations (4 homogeneous equations of 4 unknowns) with matrix

$$\mathbf{F} = \begin{bmatrix} p_{11} & p_{12} & -1 & 0 \\ p_{21} & p_{22} & 0 & -1 \\ p_{31} & p_{32} & 0 & 0 \\ p_{41} & p_{42} & 0 & 0 \end{bmatrix}. \quad (30)$$

Eigen frequencies are the non-trivial solutions of the above sets of equations with matrix Equation (31). Calculation of these solutions goes in a standard way

$$\mathrm{Det}\,\mathbf{F} = 0\,. \tag{31}$$

Frequency determinant $\det \mathbf{F} = f(\Omega)$ is a function of Ω. Eigen frequencies of system are solutions of nonlinear algebraic equation $f(\Omega) = 0$ (i.e., intersections with axis). The number of solutions is indefinite. Appropriate solutions are found only in the interval $(\Omega_{\min}, \Omega_{\max})$.

Process of calculation of Ω is done in several steps (see Figure 3).

It is sufficient to take $\Delta\Omega$ of the order of one hundredth to a thousandth of the length $(\Omega_{\max} - \Omega_{\min})$ of the frequency interval.

If the signum of the function $f(\Omega)$ changes between $\Omega_{\min} + j \cdot \Delta\Omega$ and $\Omega_{\min} + (j+1) \cdot \Delta\Omega (j = 0,1,2\ldots)$, $\Delta\Omega$ must be reduced to $\frac{1}{10}$ and the process is repeated with the value $\frac{\Delta\Omega}{10}$. The calculation is finished when $\left|\frac{f}{f_{\min}}\right| < \varepsilon < 1$.

9 Conclusions

Modelling of the rotating shaft is influenced by many factors, such as workspace of universal joints [3], vibration noise [4], flexibility of a shaft [5], increasing rotation velocity [6].

More frequently separated movements of the shaft are studied (see [1, 7-9]). Sinitsin and Shestakov [10] present a comprehensive analysis of the angular and linear accelerations of moving elements (shafts, gears) by wireless acceleration sensor of moving elements. The combined motions are presented in paper [2].

A procedure for vibration analysis of the device, based on measured data in simulated operating modes in mechanisms, is studied in [11]. In [12], these new trends in torsional vibration calculation for various vehicles are briefly described, with attention paid not only to practical use but above all to how and to what extent these themes should be presented to students.

In the paper, the following problems are presented:

- Calculation the vector of state in every part of the shaft based on known physical characteristics;
- Calculation of transfer matrix **P** and all the other amplitude-frequency characteristics of the state quantity are done in Octave 4.2.0;
- Assessing the resistance of the shaft to transverse oscillation during the design by application of the transfer matrix method.

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