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FAST ANALYTICAL DESIGN OF MAXIMALLY FLAT NOTCH FIR FILTERS

A novel fast analytical design procedure for the maximally flat notch FIR filters is introduced. The closed form solution provides recursive evaluation of the impulse response coefficients of the filter. The discrete nature of the notch frequency is emphasized. One design example is included to demonstrate the efficiency of the presented approach.

Keywords: *Notch filters, Maximally Flat, Analytical Design*

1. Introduction

The narrow band digital filters are widely used in digital signal processing. While narrow band-pass filters find their application in the detection of signals, the narrow bandstop filters are frequently used in order to remove a single frequency component from the spectrum of the signal. The narrow bandstop filters are usually called notch filters. In our paper we primarily deal with notch filters but we keep in mind the close relation between these two types of narrow band filters. The design of digital notch IIR filters is rather simple. These filters are frequently used despite of their infinite impulse and step responses, which can produce spurious signal components that are unwanted in various applications. The notch IIR filters consist of an abridged all-pass second-order section that allows independent tuning of the notch frequency $\omega_m T$ and the 3-dB attenuation bandwidth [3]. The main drawback usually emphasized in connection with FIR filters is the higher number of coefficients compared to their IIR counterparts. However, this argument is weakened continuously due to the tremendous advance in DSP and FPGA technology. The decisive advantages of FIR filters are their constant group delay and superior time response [8]. Thus the implementation of FIR filters with one hundred coefficients has a practical impact in numerous applications. A few analytical procedures for the design of linear phase notch FIR filters have recently become available [5]. The methods, which lead to feasible filters, are generally derived by iterative approximation techniques or by non-iterative, but still numerical procedures, e.g. the window technique. In our paper we are concerned with completely analytical design of maximally flat (MF) notch FIR filters. We introduce the degree formula, which relates the degree of the generating polynomial, the length of the filter, the notch frequency, the width of the notchband and the attenuation in the passbands. We derive the differential equation for the generating polynomial of the filter. Based on the expansion of the generating polynomial into the Chebyshev polynomials, the recurrent formula for the direct computation of the impulse response coefficients is derived. Consequently, the FFT algorithm usually required in the analytical design of narrow band FIR filters is avoided. The proposed design procedure is recursive one. It does not require any FFT algorithm or any iterative technique.

2. Polynomial Approximation, Zero Phase Transfer Function

Here and in the following we use the independent transformed variable w [6] related to the digital domain by

$$\omega = \frac{1}{2}(z + z^{-1})\big|_{z=e^{j\omega T}} = cos\omega T. \qquad (1)$$

We denote $H(z)$ the transfer function of a notch FIR filter with the impulse response h(m) of the length N as

$$H(z) = \sum_{m=0}^{N-1} h(m) z^{-m}. \qquad (2)$$

Assuming an odd length $N = 2n+1$ and even symmetry of the impulse response

$$a(0) = h(n),\ a(m) = 2h(n \pm m),\ m = 1 \ldots n \qquad (3)$$

we can write the transfer function of the notch FIR filter

$$H(z) = z^{-n}\left[a(0) + \sum_{m=1}^{n} a(m)\, T_m(w)\right] \qquad (4)$$

where $T_m(w)$ is Chebyshev polynomial of the first kind. The frequency response of the filter $H(e^{j\omega T})$ can be expressed by the zero phase transfer function $Q(w)$

$$H(e^{j\omega T}) = e^{-jn\omega T}\, Q(\cos\omega T) = z^{-n}\, Q(w)\big|_{z=e^{j\omega T}} \qquad (5)$$

For $w = 0.5\,(z + z^{-1})\big|_{z=e^{j\omega T}} = \cos\omega T$ the zero phase transfer function $Q(w)$ represents a polynomial of the real variable w.

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It reduces to a real valued frequency response of the zero-phase FIR filter. The zero phase transfer function $Q(w)$ of the narrow bandpass FIR filter is formed by the generating polynomial $A_{p,q}(w)$ while the zero phase transfer function $Q_A(w)$ of the notch FIR filter is

$$Q_A(w) = 1 - A_{p,q}(w). \tag{6}$$

3. Maximally Flat Notch FIR Filter

For the design of MF notch FIR filter we propose the generating polynomial $A_{p,q}(w)$ of the MF narrow bandpass FIR filter introduced in [7]

$$A_{p,q}(w) = C(1 - w)^p (1 + w)^q. \tag{7}$$

The notation $A_{p,q}(w)$ emphasizes that p counts multiplicity of zeros at $w = 1$ and q corresponds to multiplicity of zeros at $w = -1$. Forming the derivative of the polynomial

$$\frac{dA_{p,q}(w)}{\mathrm{dw}} = -Cp(1 - w)^{p-1}(1 + w)^q + + Cq(1 - w)^p(1 + w)^{q-1} \tag{8}$$

and by simple manipulation of (7)

$$(1 - w)(1 + w)\frac{dA_{p,q}(w)}{dw} = = -p(1 + w)A_{p,q}(w) + q(1 - w)A_{p,q}(w) \tag{9}$$

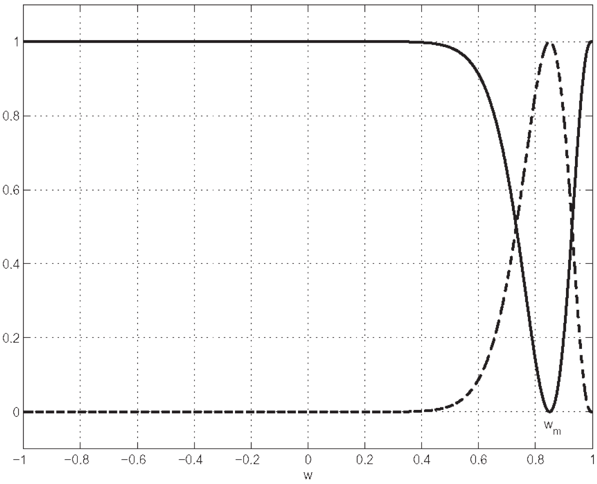

Fig. 1. Generating polynomial $A_{3,37}(w)$ (dashed) and the zero phase transfer function $Q_A(w) = 1 - A_{3,37}(w)$ of the MF notch FIR filter with extremal value for $w_m = (37 - 3)/(37 + 3) = 0.85$

we arrive at the differential equation for the generating polynomial $A_{p,q}(w)$

$$(1 - w^2)\frac{dA_{p,q}(w)}{dw} + [p - q + (p + q)w]A_{p,q}(w) = 0\,. \tag{10}$$

The differential equation (10) for the polynomial $A_{p,q}(w)$ forms a completely new concept in digital filter design as it provides the recursive evaluation of the impulse response coefficients of the filter described in Section 5. The normalization of the generating polynomial $A_{p,q}(w)$ constraints $A_{p,q}(w_m) = 1$ where w_m is the position of the maximum of the generating polynomial $A_{p,q}(w)$ as illustrated in Fig. 1. The normalization of the generating polynomial $A_{p,q}(w)$ results in

$$A_{p,q}(w) = \left[\frac{p+q}{2p}(1 - w)\right]^p \left[\frac{p+q}{2q}(1 + w)\right]^q \tag{11}$$

The polynomial

$$Q_A(w) = 1 - A_{p,q}(w) = 1 - \left[\frac{p+q}{2p}(1 - w)\right]^p \left[\frac{p+q}{2q}(1 + w)\right]^q \tag{12}$$

represents the real-valued zero phase transfer function of the MF notch FIR filter of the real variable $w = \cos \omega T$. For illustration, the zero phase transfer function of the MF notch FIR filter $Q_A(w) = 1 - A_{3,37}(w)$ is shown in Fig. 1. The corresponding amplitude frequency response $20 \log |H(e^{j\omega T})|$ [dB]

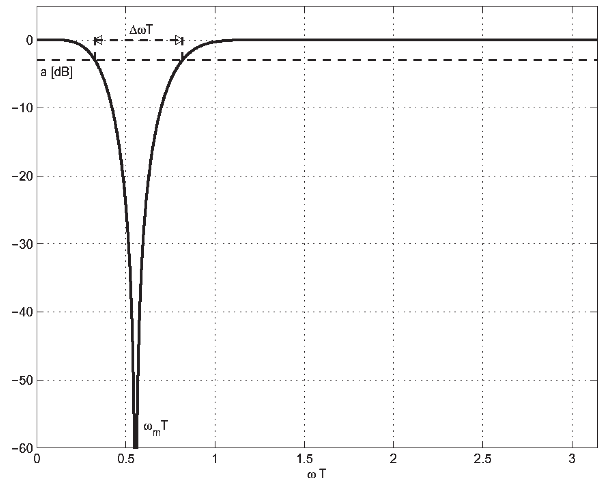

Fig. 2. Amplitude frequency response $20 \log |H(e^{j\omega T})|$ [dB] based on the generating polynomial $Q_A(w) = 1 - A_{3,37}(w)$ from Fig. 1. The parameters are $\omega_m T = 0.1766\,\pi$ and $\Delta\omega T = 0.1555\,\pi$ for the standard attenuation $a = -3.0103$ dB

is shown in Fig. 2. The transfer function of the MF notch FIR filter is

$$H(z) = \sum_{m=0}^{N-1} h(m)\, z^{-m} = z^{-n}\,(1 - A(p,q)(w)). \tag{13}$$

4. Notch Frequency and the Degree of the Maximally Flat Notch FIR Filter

The notch frequency $\omega_m T$ is derived from the minimum value of the zero phase transfer function $Q_A(w)$ (12) as

$$w_m = \cos \omega_m T = \frac{q-p}{q+p}. \qquad (14)$$

The notch frequency $\omega_m T$ of the MF notch FIR filter is given from (14) by the integer values p and q exclusively. It is obvious that for the specified filter length $N = 2(p + q) + 1$, exactly $p + q - 1$ discrete notch frequencies $\omega_m T$ are available. From the symmetrical case $n/2 = p = q$ the degree equation

$$n \geq \frac{\log(1 - 10^{0.05a|dB|})}{\log \cos \frac{\Delta\omega T}{2}} \qquad (15)$$

can be derived. The relations for the integer values p, q read as follows

$$p = \left[n \sin^2\left(\frac{\omega_m T}{2}\right) \right], \; q = \left[n \cos^2\left(\frac{\omega_m T}{2}\right) \right]. \qquad (16)$$

The brackets [ ] in (16) denote the rounding operation.

5. Impulse Response Coefficients of the Maximally Flat FIR Filter

We can express the generating polynomial $A_{p,q}(w)$ of the degree $n = p + q$ as the sum of Chebyshev polynomials of the first kind $T_m(w)$

$$A_{p,q}(w) = \sum_{m=0}^{n} a(m) T_m(w). \qquad (17)$$

The coefficients $a(m)$ define the impulse response $h(m)$ (3) of the length $N = 2(p + q) + 1$. Assuming the generating polynomial $A_{p,q}(w)$ of the MF narrow bandpass FIR filter in the sum (17) we can write

$$(1 - w^2)\frac{dA_{p,q}(w)}{dw} = \sum_{m=1}^{n} a(m)(1 - w^2)\frac{dT_m(w)}{dw} =$$

$$= \sum_{m=1}^{n} a(m)\frac{m}{2}[T_{m-1}(w) - T_{m+1}(w)]. \qquad (18)$$

By introducing (17) and (18) into the differential equation (10) and using the recursive formula for Chebyshev polynomials

$$T_{m+1}(w) = 2\, w T_m(w) - T_{m-1}(w) \qquad (19)$$

we get the identity

$$\sum_{m=1}^{n} a(m)\frac{m}{2}[T_{m-1}(w) - T_{m+1}(w)] + (p - q)a(0) +$$

$$+ \sum_{m=1}^{n} a(m)(p - q)T_m(w) + (p + q)a(0)w + \qquad (20)$$

$$+ \sum_{m=1}^{n} a(m)(p + q)\frac{1}{2}[T_{m-1}(w) + T_{m+1}(w)] = 0.$$

By iterating eq. (20) we have deduced a simple recursive algorithm for the evaluation of the coefficients $a(m)$ of the generating polynomial $A_{p,q}(w)$ of the MF narrow bandpass FIR filter. The recursive algorithm is presented in Table 1. The coefficients $h(m)$ of the impulse response of the MF notch FIR filter are obtained from the coefficients $a(m)$ of the MF narrow bandpass FIR filter as follows

$$h(n) = 1 - a(0), \; h(n \pm m) = -\frac{a(m)}{2}, \; m = 1 \ldots n. \qquad (21)$$

6. Design of the Maximally Flat Notch FIR Filter

The goal of the MF notch FIR filter design is to find the two integer values p and q in order to satisfy the filter specification as precisely as possible. The design procedure is as follows:

1. Specify the notch frequency $\omega_m T$, maximal width of the notchband $\Delta\omega T$ and the attenuation in the passbands a [dB] as demonstrated in Fig. 2.
2. Calculate the minimum degree n (15) required to satisfy the filter specification.
3. Calculate the integer values p and q (16).
4. Check the notch frequency (14) for the obtained integer values p, q .
5. Evaluate the coefficients $a(m)$ of the generating polynomial $A_{p,q}(w)$ recursively (Table 1).
6. Evaluate the coefficients of the impulse response $h(m)$ of the MF notch FIR filter (21).

Recursive algorithm for evaluation of the coefficients $a(m)$ Tab. 1

given	p, q
initialization	$n = p + q$ $a(n+1) = 0$
body (*for* $k = n + 1$ *to* 3)	$a(k-2) = -\dfrac{(n+k)a(k) + 2(2p-n)a(k-1)}{n+2-k}$
(*end loop on* k)	$a(0) = -\dfrac{(n+2)a(2) + 2(2p-n)a(1)}{2n}$

It is worth of noting that a substantial part of coefficients of the impulse response $h(m)$ of the MF notch FIR filter has negligible values. From this

Impulse Response Coefficients Table 2

m		h(m)	m		h(m)
14	74	-0.000002	30	58	0.012289
15	73	-0.000003	31	57	0.002278
16	72	0.000000	32	56	-0.019427
17	71	0.000018	33	55	-0.027483
18	70	0.000037	34	54	-0.003357
19	69	0.000010	35	53	0.042804
20	68	-0.000111	36	52	0.048063
21	67	-0.000245	37	51	-0.009353
22	66	-0.000101	38	50	-0.075616
23	65	0.000537	39	49	-0.065324
24	64	0.001173	40	48	0.029196
25	63	0.000480	41	47	0.106554
26	62	-0.002149	42	46	0.068113
27	61	-0.004302	43	45	-0.053105
28	60	-0.001388	44		0.880514
29	59	0.007135			

fact follows the possible large abbreviation of the impulse response of the MF notch FIR filter by the rectangular windowing without significant deterioration of the frequency properties of the filter as emphasized in [7].

7. Example of the Design of the Maximally Flat Notch FIR Filter

Design the MF notch FIR filter specified by $\omega_m T = 0.35\ \pi$ and $\Delta\omega T = 0.15\ \pi$ for $a = -3.0103$ dB.

Using our design procedure we get $n = [43.8256] \rightarrow 44$ (15), $p = [11.9644] \rightarrow 12$ and $q = [31.8610] \rightarrow 32$ (16). The filter length is $N = 89$ coefficients. The actual filter parameters are $\omega_m T = 0.3498\ \pi$ and $\Delta\omega T = 0.1496\ \pi$. The attenuation at the frequency

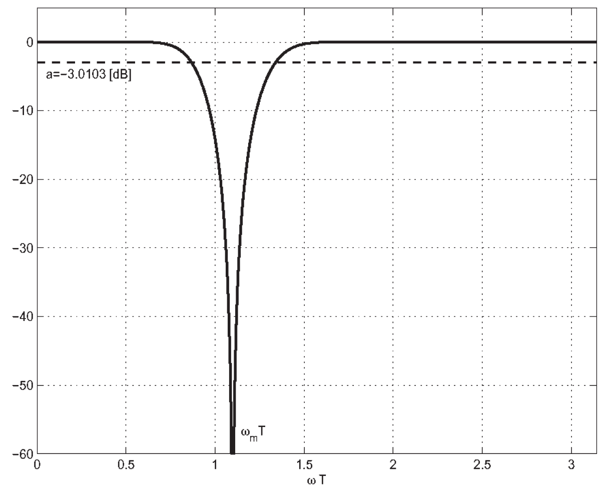

Fig. 3. Amplitude frequency response 20 log $|H(e^{j\omega T})|$ [dB] based on the zero phase transfer function $Q(w) = 1 - A_{12,32}(w)$

$0.3\ \pi$ amounts -168 dB. The coefficients $a(m)$ were evaluated recursively (Table 1). The coefficients of the impulse response $h(m)$ of the MF notch FIR filter were evaluated by (21). Because $|h(m)| < 10^{-6}$ for $0 < m < 14$ and $m > 74$, only the 71 central coefficients of the impulse response $h(m)$ are summarized in Table 2. The amplitude frequency response 20 log $|H(e^{j\omega T})|$ [dB] of the MF notch FIR filter is shown in Fig. 3.

8. Conclusions

Novel fast analytical design procedure for the design of maximally flat notch FIR filters was introduced. The closed form solution provides recursive evaluation of the impulse response of the filter. One example demonstrated the efficiency of the design procedure.

9. Acknowledgements

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