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COMMUNICATIONS

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WAITING TIME OPTIMIZATION WITH IP-SOLVER

This paper deals with two different public transport problems, in which the same phenomenon of waiting time occurs. In the past, both the problems were solved in the same way, which included rearrangement of the original problems to simpler max-min problems. These simplified approaches were used due to that time state of optimization software. In connection with building new computational laboratories equipped with new optimization environment, we want to come back to the more precise original quadratic models of the problems and explore new possibilities in obtaining the optimal solution of the original problems. We have done an analysis of those quadratic programming problems, worked out a linearized model and completed the computational study to compare the max-min and quadratic approaches.

1. Introduction

A series of mathematical programming models of transportation problems were formulated and solved in the several last decades to obtain solution of the transportation problems. In those models, regular distribution of time intervals was taken as a quality criterion of searched solutions, even if the original objective was to minimize the total waiting time of passengers or cars in traffic flows. This original criterion was replaced by the criterion of regularity to preserve linearity of the processed mathematical models. This simplified approach was used because of that time state of computation technique, which did not allow complying with nonlinear or large linear problems. Furthermore, the criterion of regularity was often simplified to min-max or max-min objective function and so, only the worst time interval of the solved problem was improved by the associated optimization process.

This way, the obtained results were far from the optimal ones in many cases, even if an exact method was used to solve the associated linear programming problem. In this paper, we present two transportation problems with the original and surrogate objective functions and compare the results obtained by solving a simplified model with the max-min criterion and a more precise and larger model, which respects the quadratic criterion. This comparison including the inevitable large problem solving is enabled by exploitation of optimization environment called XPRESS-IVE. Abilities of this tool are also studied in this paper in connection with the necessity to solve much larger linear problems to comply with the quadratic criteria.

2. Max-min approach to the signal plan for light-controlled crossing

Let us consider that a set I represents a set of traffic flows at a crossing. Each traffic flow $i \in I$ is characterized by intensity f_i, i.e. number of vehicles that enter the crossing per time unit, and the saturated intensity f_i^s of the flow, which is a maximum number of vehicles that can leave the crossing per time unit. Let τ_i be the standard for a minimum duration of green light for the flow i at the crossing.

Let $K = \{F_1, F_2, \dots F_r\}$ be the set of r phases at the crossing. A phase F_k is a set of non-collision flows that can have simultaneously green light at the crossing. We assume that the phases follow in the order given by their indices and that the flow of green light period for all phases falls into the interval $\langle 0, t_{max} \rangle$. Let m_{ij} be the minimal interval between two successive collision flows from different phases and let t_{max} be the time of crossing period duration.

The natural objective is to design a signal plan so that the total waiting time of all relevant participants is minimal. Let us realize what the waiting time is for a flow i with the intensities f_i and f_i^s, when duration of the red light is denoted as t_i^r and duration of the green light is denoted as t_i^g.

Figure 1 depicts the dependence of a number of waiting vehicles on the time during one period of a signal plan of a crossing. The shadow area in the figure corresponds with the total waiting time of all participants of the flow i entering the crossing during the period t_{max}.

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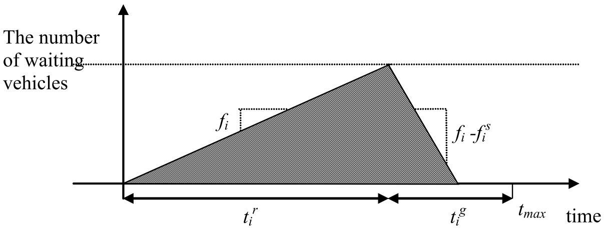

Fig. 1 The waiting time of the flow i during period t_{max}

The total waiting time of all participants of the flow i during the period t_{max} is:

$$0.5 * (t_i^r)^2 * f_i + 0.5 * (t_i^r)^2 * (f_i)^2/(f_i^s - f_i) = \\ = 0.5 * (t_i^r)^2 * f_i^s/(f_i^s - f_i) \quad (1)$$

To build a model of the problem, we introduce the variable x_i as the starting time of the green signal of each traffic flow i during one period and the variable y_i as the ending time of the green signal of traffic flow i during one period. Due to simplification, we denote the value of expression (2) as c_i.

$$0.5 * f_i * f_i^s/(f_i^s - f_i). \quad (2)$$

To model the duration t_i^r of the red light, we introduce the auxiliary variable u_i, and then the model of the original problem can be stated as follows:

$$\textit{Minimize} \sum_{i \in I} c_i * (u_i)^2 \quad (3)$$

$$\textit{Subject to} \quad t_{max} - y_i + x_i = u_i \quad \textit{for} \ i \in I \quad (4)$$

$$y_i - x_i \geq \left(\frac{f_i t_{max}}{f_i^s} + 1\right) \quad \textit{for} \ i \in I \quad (5)$$

$$y_i - x_i \geq \tau_i \quad \textit{for} \ i \in I \quad (6)$$

$$x_j - y_i \geq m_{ij} \quad \textit{for} \ k = 1, ..., r-1 \ i \in F_k, \ j \in F_{k+1} \quad (7)$$

$$x_j - y_i \geq m_{ij} - t_{max} \quad \textit{for} \ i \in F_r, j \in F_1 \quad (8)$$

$$x_i \in Z^+ \quad \textit{for} \ i \in I \quad (9)$$

$$y_i \in Z^+ \quad \textit{for} \ i \in I \quad (10)$$

$$u_i \geq 0 \quad \textit{for} \ i \in I \quad (11)$$

The constraints (4) are link-up constraints connecting starting and ending times of the green light period with the associated length u_i of the red light period.

The constraints (5) assure that time of the green signal for the traffic flow i is at least as long as the crossing time for the passing of all incoming vehicles. The constraints (6) assure that time of the green signal for the traffic flow i is at least as long as the standard time (if the crossing time for the passing of all incoming vehicles is negligible).

Constraints (7) and (8) assure that the gap between the ending time and starting time of two collision traffic flows from two consecutive phases is greater or equal to the minimal interval between these two flows. Constraints (8) assure this situation for the traffic flows between the last and the first phase.

Unfortunately, the model (3)-(11) is non-linear because of the objective function (3). This constituted serious obstacle in the period, when the first attempt at the problem solving was done. That is why the non-linear problem was substituted by linear one. The substitution was in the following way [1]. The objective function corresponding with the total waiting time during the period t_{max} was abandoned and replaced by a demand that the minimal relative reserve of the relevant traffic flows should be as high as possible. The relative reserve of the flow i with time of the green signal t_i^g is defined by the ratio (12).

$$\frac{t_i^g}{\left(\frac{f_i t_{max}}{f_i^s} + 1\right)} \quad (12)$$

To model this rearranged problem, we introduce a variable u as a lower bound on each relative reserve of all relevant flows. Now, making use of the above-mentioned variables x_i and y_i, the new problem can be described as follows.

$$\textit{Minimize} \ u \quad (13)$$

$$\textit{Subject to} \quad y_i - x_i \geq u\left(\frac{f_i t_{\max}}{f_i^s} + 1\right) \quad \textit{for} \ i \in I \quad (14)$$

$$y_i - x_i \geq \tau_i \quad \textit{for} \ i \in I \quad (15)$$

$$x_j - y_i \geq m_{ij} \quad \textit{for} \ k = 1, ..., r-1 \ i \in F_k, \ j \in F_{k+1} \quad (16)$$

$$x_j - y_i \geq m_{ij} - t_{max} \quad \textit{for} \ i \in F_r, j \in F_1 \quad (17)$$

$$x_i \in Z^+ \quad \textit{for} \ i \in I \quad (18)$$

$$y_i \in Z^+ \quad \textit{for} \ i \in I \quad (19)$$

$$u_i \geq 0 \quad (20)$$

Assuming that $u \geq 1$, the constraints (14) assure that the time of the green signal for the traffic flow i is at least as large as the crossing time for the passing of all incoming vehicles. Relative reserve of the traffic flow i must be greater or equal to the lower bound u. The other constraints have the same meaning as constraints (6)-(8) respectively.

Comparing the two models, we have to admit that they are not equivalent, which implies that the result of the second problem solution need not optimize the original objective function. In the computational study we point out these differences and demonstrate their consequences.

3. Max-min approach to the arrival time coordination in public transport

Let us consider that a set I represents a set of n vehicle arrivals at an observed stop in a given period. Let t_i denote the time of the arrival i. This arrival time can be shifted from a time a_i, which denotes the earliest arrival time of the associated vehicle, to the time $a_i + c_i$, which denotes the last arrival of the vehicle. The period c_i is a maximal shift from the earliest arrival time of the vehicle.

Let t_0 be a fixed time of the first arrival and t_n be a fixed time of the last vehicle arrival. It is assumed that passengers come to the observed stop with an average intensity f. The objective is to move the times t_i for $i = 1, ..., n-1$ so that the total waiting time of passengers is minimal.

Figure 2 depicts the dependence of waiting passengers on the time during the period $\langle t_0, t_n \rangle$. The shadow area in the figure corresponds with the total waiting time of all passengers visiting the stop during the considered period.

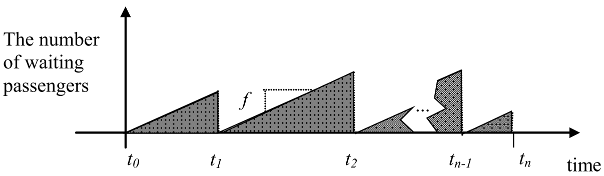

Fig. 2 The waiting time of passengers during period $\langle t_0, t_n \rangle$

It follows that the total waiting time of all considered passengers during the period $\langle t_0, t_n \rangle$ is:

$$\sum_{i=1}^{n} 0.5 * f * (t_i - t_{i-1})^2 \qquad (21)$$

To simplify the following model, we introduce an auxiliary variable ui as the maximal waiting time between the arrivals t_{i-1} and t_i for $i = 1, ..., n$. We also introduce a variable x_i, for $i = 1, ..., n-1$, which corresponds with a shift of the arrival time t_i versus time a_i. Then the model of this original problem can be stated as follows:

$$\textit{Minimize}\ \ 0.5 * f * \sum_{i=1}^{n}\sum_{i \in 1} (u_i)^2 \qquad (22)$$

$$\textit{Subject to}\ \ x_1 + a_1 - t_0 + u_1 \qquad (23)$$

$$x_i + a_i - x_{i-1} - a_{i-1} = u_1 \quad \textit{for}\ \ i = 2, ..., n-1 \qquad (24)$$

$$t_n - x_{n-1} - a_{n-1} = u_n \qquad (25)$$

$$x_i \le c_i \quad \textit{for} \quad i = 1, ..., n-1 \qquad (26)$$

$$x_i \ge 0 \quad \textit{for} \quad i = 1, ..., n-1 \qquad (27)$$

$$u_i \ge 0 \quad \textit{for} \quad i = 1, ..., n \qquad (28)$$

Similarly as in the previous case of waiting time at the crossing also in this case [1] there was no smart tool at disposal to solve the quadratic problem (23)–(28). That was why the approach of maximization of the shortest period between consecutive arrivals was used. The variables x_i, for $i = 1, ..., n-1$ were introduced as above and the variable y was used as the lower bound of periods between pairs of consecutive arrivals. Then, the following linear model was obtained:

$$\textit{Minimize}\ y \qquad (29)$$

$$\textit{Subject to}\ \ x_1 + a_1 - t_0 \ge y \qquad (30)$$

$$x_i + a_i - x_{i-1} - a_{i-1} \ge y \quad \textit{for}\ \ i = 2, ..., n-1 \qquad (31)$$

$$t_n - x_{n-1} - a_{n-1} \ge y \qquad (32)$$

$$x_i \le c_i \quad \textit{for} \quad i = 1, ..., n-1 \qquad (33)$$

$$x_i \ge 0 \quad \textit{for} \quad i = 1, ..., n-1 \qquad (34)$$

$$y \ge 0 \qquad (35)$$

Constraints (30), (31) and (32) assure that any time gap between arrival times of two consecutive arrivals must be greater or equal to the lower bound y. Constraints (33) assure that the time shift of the arrival time i is not greater than the maximal value of the shift for the arrival time.

4. Linearization of quadratic criteria

As mentioned before, the more precise original models with the waiting times expressions included into their objective functions had to be abandoned due to non-linearity even when the way of linearization had been known [2], [3]. The reason was that the linearized model after rearrangement becomes too large to be solved by the past tools.

In this paper we focus on answering the question whether the new techniques implemented in today's optimization tools are able to overcome the former obstacles. Further we will show the way of linearization, which can be used to replace objective functions (3) and (22) by linear expressions almost without loss of accuracy.

We have to realize several next properties of the processed non-linear models. First, both considered objective functions are separable. It means that each non-linearity included in summation depends only on one variable, whose value is bounded from lower and upper sides by the values 0 and u_i^{max} respectively. Second, the objective functions are convex and their minimal value is searched for. Third, the time values in the transportation problems are given in some integer units, e.g. seconds or minutes. It means that one unit is a maximal accuracy, which is necessary to take into account. It follows that the quadratic function $(u_i)^2$ can be replaced by a piecewise linear function without loss of accuracy as shown in Figure 3.

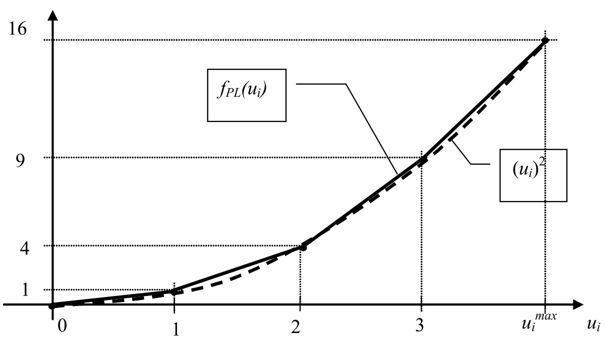

Fig. 3 The quadratic function and its approximation by a piecewise linear function $f_{PL}(u_i)$

To replace the non-linear item $(u_i)^2$ by a piecewise linear function in the range $\langle 0, u_i^{max} \rangle$, where u_i^{max} is integer, we introduce a set of auxiliary variables z_{ij}, where $0 \leq z_{ij} \leq 1$ for $j = 1, ..., u_i^{max}$. Then, the relation between variables u_i and z_{ij} can be expressed by equation (36).

$$u_i = \sum_{j=1}^{u_i^{max}} z_{ij} \tag{36}$$

The non-linear item $(u_i)^2$ can be replaced by the right-hand-side of equation (37).

$$(u_i)^2 = \sum_{j=1}^{u_i^{max}} (2 * j - 1) z_{ij} \tag{37}$$

In a common case when this way of linearization is used it is necessary to assume that $z_{ij+1} = 0$ follows from $z_{ij} < 1$. Nevertheless, the assumption of convexity of the minimized objective function approves this implication.

Now, models (2)–(11) and (22)–(28) can be linearized by introducing a series of variables $z_{ij} \geq 0, j = 1, ... u_i^{max}$ for each non-linearity $(u_i)^2$. The quadratic items in the objective function must be replaced by a linear expression according to equation (37) and link-up constraint (36) must be added to the model for each u_i. Furthermore, each model must be enlarged by constraints (38).

$$z_{ij} \leq 1 \quad for \quad i \in I, j = 1, ..., u_i^{max} \tag{38}$$

This way, the models become linear and linear-programming solvers programmers can solve the associated problems. Nevertheless, we have to note that the number of auxiliary variables z_{ij} can be a considerably large number in some cases. The number is equal to the value of expression (39).

$$\sum_{i=1}^{n} u_i^{max} \tag{39}$$

5. Case study by XPRESS-IVE

To perform the computation of the original problems with the waiting time and also the derived max-min problems, we used the general optimization software environment XPRESS-IVE for our study [4], [5]. This software system includes the branch-and-cut method and also enables solution of large linear programming problems. The software is equipped with the programming language *Mosel*, which can be used for both the input of a model and writing of input and output procedures. The experiments were performed on a personal computer equipped with Intel Core 2 Duo E6850 with parameters 3 GHz and 3.5 GB RAM.

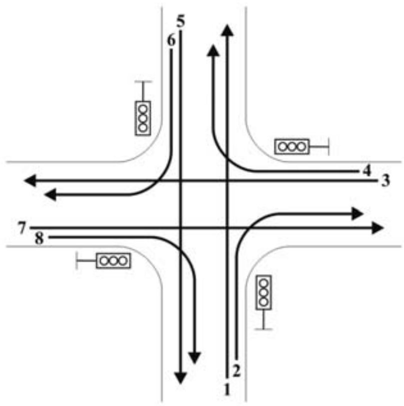

Fig. 4 Signal plan example

To verify the method we formulated an instance for each problem. The instance of a signal-plan determination for a light-controlled crossing problem consists of 8 traffic flows which are divided into two phases. The first phase consists of flows 1, 2, 5 and 6, and the second phase consists of flows 3, 4, 7 and 8. The situation with traffic flows is described in Fig. 4. The values of

Signal plan example – f_i, f_i^s, τ_i Table 1

i	f_i	f_i^s	τ_i [s]	F_1	F_2
1	0.1	0.3	10	1	
2	0.2	0.5	10	1	
3	0.15	0.4	10		1
4	0.2	0.6	10		1
5	0.1	0.3	10	1	
6	0.15	0.35	10	1	
7	0.25	0.6	10		1
8	0.15	0.4	10		1

flow intensities f_i, saturated flow intensities f_i^s, standard minimum duration τ_i of the green light for the flow i and assignment of flows to phases are reported in Table 1. Table 2 contains the values of the minimal time period m_{ij} between two successive collision flows from different phases. The value of time of the crossing period t_{max} was set to 150 seconds.

Signal plan example – m_{ij} Table 2

m_{ij}	1	2	3	4	5	6	7	8
1	–	–	8	8	–	–	8	–
2	–	–	–	–	–	–	10	–
3	8	–	–	–	8	8	–	–
4	10	–	–	–	–	–	–	–
5	–	–	8	–	–	–	8	8
6	–	–	10	–	–	–	–	–
7	8	8	–	–	8	–	–	–
8	–	–	–	–	10	–	–	–

Results obtained by software environment XPRESS-IVE are presented in Table 3. The column "Max-min" contains resulting lengths of the green signal for flow i obtained by max-min approach (13)–(20) and the column "Quadratic" contains the lengths of the green signal for flow i obtained by the linearization of quadratic criteria (2)–(11),(37),(38). "Waiting time" denotes the value of the total waiting time in vehicle-seconds, "Row" denotes the number of structural constraints of the model and "Columns" denotes the number of used variables. The computational time in both cases

Solutions for signal plan example obtained by max-min approach and linearized quadratic criteria Table 3

i	Max-min	Quadratic
	$y_i - x_i$	$y_i - x_i$
1	60	51
2	65	61
3	61	63
4	55	81
5	65	51
6	70	66
7	68	74
8	62	81
Rows	48	56
Columns	17	1216
Waiting Time	698 [vs]	573 [vs]

The instance of the arrival time coordination problem consists of 8 vehicle arrivals which can be shifted. The description of the instance and the associated solution are given in Fig. 5 and Table 4. The column a_i denotes the earliest possible arrival time of an associated vehicle, c_i denotes the maximal shift of arrival of the vehicle. We used the value of 10 for the intensity f of the passengers coming to the stop.

The column "Max-min" contains resulting lengths of intervals between two successive arrivals obtained by the max-min approach (29)–(35). The column denoted as "Quadratic" contains the lengths obtained by the second method. "Waiting time" denotes the value of the total waiting time in person-seconds, "Rows" denotes the number of structural constraints of the model and "Columns" denotes the number of used variables. The computational time in both cases was also less than 1 second.

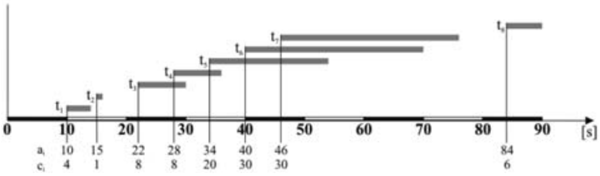

Fig. 5 Instance of the arrival time coordination problem

Solutions obtained using max-min approach and linearized quadratic criteria Table 4

i	a_i	c_i	Max-min	Quadratic
			$t_i - t_{i-1}$	$t_i - t_{i-1}$
0	0	0	–	–
1	10	4	10	10
2	15	1	6	6
3	22	8	6	10
4	28	8	6	10
5	34	20	6	12
6	40	30	6	12
7	46	30	6	12
8	84	6	38	12
9	90	0	6	6
Rows			17	25
Columns			9	359
Waiting Time			16760 [ps]	9480 [ps]

6. Conclusions

We renewed a solving approach to the public transport problems which originally included the waiting time in their objectives; nevertheless they had been solved by much simpler max-min method. Our contribution to the problem solving consists in complying with the original non-linear objective function which expresses the time lost by waiting passengers. To solve the problems with the original objective function, we applied a piecewise linear approximation of the quadratic items and made use of special properties of optimization environment XPRESS-IVE to solve the resulting large linear problems. We implemented both former and latter method to be able to compare the resulting optimal solutions, the sizes of processed models and the computational times necessary for obtaining optimal solutions. We found that even if piecewise linear models are much larger than the previously used max-min models, the computational times increased negligibly. With regard to the quality of optimal solutions, the comparison shows that the solutions obtained by the renewed approach are much better than those obtained by the former max-min approach.

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