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# GROUPS WITH THE INFINITE NON-QUASICENTRAL NODAL SUBGROUP

We present some properties of the groups with the infinite non-quasicentral periodic nodal subgroup. Our main results are formulated in Theorem 1, 2 and Theorem 3, 4, 2000 Mathematics Subject Classification: 20F22, 20F25

**Keywords:** A group, a subgroup, a commutator of a group, a locally graded group, p-quasicyclic group, a direct and semi-direct product of groups, an extension of a group.

The description of groups defined by the systems of their subgroups was first described in the papers of Chernikov and Kurosh (RN – groups, [1]). Chernikov dealt with an extension of the direct product of the finite number of the quasicyclic groups by the finite abelian group (u – groups, [2]), with the infinite non-abelian groups whose arbitrary infinite subgroup is the normal subgroup of the whole group (INH – groups, [2]). Subbotin studied the groups G in which every subgroup from commutator G' is a normal subgroup of G (KI–groups, [3]). Tomanek, L. studied the IAN and the IANA groups, Definition 1, (IAN groups, [4]). This definition was given to the author by Chernikov. In this paper we describe IAN and IANA groups with the infinite non-quasicentral periodic nodal subgroup.

We use standard designations of terminology where:  $M \times N$ is the direct product of the groups M, N;  $\sum_{i \in I} Xi$  is the direct sum of the additive groups  $X_i$  for all  $i \in I$ ,  $M \setminus N$  is the semidirect product of the groups M, N;  $M.N = \{mn \mid m \in M, n \in N\}$ is the product of the groups M, N, G/A is the factor group of Gby A, |G:N| is the index of the subgroup N in a group G;  $\langle a \rangle$ is the cyclic group generated by the element  $a_i < a_i$ ,  $b_i$ , c > is the group generated by the elements a, b, c;  $H \le G$  where H is the subgroup of G;  $H \subseteq G$ , H is normal in G;  $[a,b] = a^{-1} b^{-1} ab$  is the commutator of the elements  $a,b \in G$ ; G' = [G,G] generated by all commutators of the elements  $a,b \in G$  is the commutator of G; [5]  $Z(p^{\infty}) = \{x \mid x^{p^n} = 1, n = 1, 2, ...\}$  is the *p*-quasicyclic group;  $C_c(A)$  is the centralizer of the subgroup A in G; C(G) is the centre of the group G;  $G \cong H$  where the groups G, H are isomorphic. The group G is the p-group if each of its elements has an order with a power of some fixed prime p [6]. The group G is a locally graded group if every finitely generated nontrivial subgroup of G contains a proper subgroup of finite index ([2] p.236). The group G is the solvable group if it includes series:  $G > G' > G^{(2)} > ... > G^{(n)} = \langle e \rangle$ . The subgroup A is a quasicentral of G if every subgroup of A is normal in G. The group G is the extension of the group G by a normal subgroup G of if  $G/N \cong H$ . The extension of the finite direct product of the quasicyclic groups by the finite abelian group are G is the extension of the quasicyclic group if G is the extension of the quasicyclic group by the finite group. G is the extension of the quasicyclic group by the finite group. G is the extension of the quasicyclic group by the finite group. G is the extension of the quasicyclic group by the finite group.

# **Definition 1.**

An infinite non-abelian G is said to be the IAN group if there exists a subgroup A of G so that every infinite subgroup of A and every infinite subgroup of G containing A is a normal subgroup of G. The group G is the IANA group if A is the abelian subgroup. The subgroup A is called the nodal subgroup.

# Definition 2.

An infinite non-abelian G is the INH group if an arbitrary infinite subgroup of G is the normal subgroup of G.

## **Definition 3.**

The group G is the Dedekind group if an arbitrary subgroup of G is the normal subgroup of G. Non-abelian Dedekind group G is called the Hamiltonian group.

# **Proposition 1.** [[2], T. 6.10]

The infinite Hamiltonian groups and the non-abelian non-Hamiltonian groups that are the finite extensions of the quasicyclic

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subgroups by the finite abelian and the finite Hamiltonian groups form the class of the solvable INH groups.

# **Proposition 2.**[[7], T.12.5.4]

The group G is the Hamiltonian group if and only if the group  $G = Q_8 \times M \times N$  where  $Q_8$  is the quaternion group, M is an elementary abelian 2- group, N is a periodic abelian group with no elements of the order 2.

## Lemma 1.

Let G be the IAN group with a nodal subgroup A. If a nodal subgroup A contains the element of the infinite order, then A is the abelian quasicentral subgroup of group G.

**Proof.** If the group A contains the element x of the infinite order, then according to Definition 2, the group A is the INH group. According to Proposition 1 A is the abelian group. Let B be an arbitrary subgroup of the group A. We shall show that  $B \subseteq G$ . If B is an infinite subgroup of G, B is admittedly a normal subgroup of G.

Let B be a finite subgroup of G. If A is the abelian group containing the element x of the infinite order, then  $B \le x \ge B \times \le x \ge A$ . Pursuant to Definition 1  $(B \times \le x \ge A) \le A$  which implies A = A. Thus A = A is the abelian quasicentral subgroup of the group A = A.

# Lemma 2.

If G is the locally graded IAN group with the nodal subgroup A, there exists a subgroup of A that is not a normal subgroup of G. Then A is a finite group or A is the extension of the quasicyclic subgroup by the finite Dedekind group.

**Proof.** Let G be the IAN group with a nodal subgroup A and let  $A_1 \le A$  where  $A_1$  is not a normal subgroup of G. Admittedly,  $A_1 \ne \le A$  is a finite subgroup of G. In agreement with Lemma 1 A is a periodic group. If A is a finite group, then this lemma is valid. Let A be an infinite periodic subgroup of G. We consider two possible cases; A is not A group, or A is A group.

Case 1. Let A not be u – group. Then choose the subgroup  $A_2$  of A where  $A_2 = A_3 \times A_4$ ,  $A_2 \cap A_1 = <e>$ , and where  $A_3$ ,  $A_4$  are the infinite cyclic groups of G. By Definition 1  $A_3 \subseteq G$ ,  $A_3 \times A_1 \subseteq G$ ,  $A_4 \subseteq G$ ,  $A_4 \times A_1 \subseteq G$ . Evidently  $(A_3 \times A_1) \cap (A_4 \times A_1) = A_1$  and furthermore  $A_1 \subseteq G$ , so it is a contradiction.

Case 2. Let A be u-group. Then put A=R. B where R is the direct product of the finite number of the quasicyclic groups, R is at the same time a divisible group, and B is the finite group where  $B\neq <e>$ . Therefore,  $A_1$  is not a normal subgroup of G; there exists a cyclic subgroup <a> of  $A_1$  that is not normal in G and where R C is a divisible group, there exists a quasicyclic subgroup C of C and furthermore C contains the subgroup C vertically C where C is an infinite subgroup of C or C or C is an infinite subgroup of C, then, by Definition 1

 $R_2 \unlhd G$ , furthermore  $(R_2 \leftthreetimes \langle a \rangle) \unlhd G$ ,  $R_1 \unlhd G$ ,  $(R_1 \lessdot a \rangle) \unlhd G$ . Evidently  $(R_2 \leftthreetimes \langle a \rangle) \cap (R_1 \lessdot a \rangle) = \langle a \rangle$  and  $\langle a \rangle \unlhd G$ . This is a contradiction.

Let  $R_2 = \langle e \rangle$ , then  $R = R_1$  is a quasicyclic group and moreover  $A/R \cong B$  where B is a finite Dedekind group. Thus A is the extension of the quasicyclic subgroup by the finite Dedekind group.  $\blacksquare$ .

#### Theorem 1.

If G is a locally graded IAN group with a nodal subgroup A, then subgroup A belongs to one of the types:

- 1. A is a finite subgroup of G;
- A is an extension of the quasicyclic subgroup by a finite Dedekind group where G'is an infinite group;
- 3. A is an infinite quasicentral periodic subgroup of G;
- 4. A is a quasicentral non-periodic abelian subgroup of G.

Proof. If A is not a quasicentral subgroup of G, then, based on Lemma 2, the subgroup A belongs to one of types 1 or 2 of this theorem. If A is a quasicentral subgroup of G, then by Lemma 1 the subgroup A belongs to one of the types 3 or 4 of this theorem.  $\blacksquare$ .

By Theorem 1 and according to the definition of IANA groups the next corollary follows.

#### Corollary 1.

If G is a locally graded IANA group with a nodal subgroup A, then subgroup A belongs to one of the types:

- 1. A is a finite abelian subgroup of G;
- 2.  $A=Z(p\infty)\times B$ , where B is a finite group;
- 3. A is an infinite quasicentral periodic abelian subgroup of G;
- 4. A is a quasicentral non-periodic abelian subgroup of G.

#### Lemma 3.

If G is the locally graded group with the infinite periodic nodal subgroup A, then the subgroup A satisfies one of the following conditions:

- 1. A is the infinite periodic Dedekind quasicentral subgroup of the group G where G/A is the abelian group;
- 2. A is the infinite periodic Dedekind quasicentral subgroup of the group G where G/A is the Hamiltonian group and G is a locally finite group;
- 3. A is not the quasicentral subgroup of G, A is an almost quasicyclic subgroup of G where G/A is the Dedekind group.

**Proof.** If G is the locally graded group with an infinite periodic nodal subgroup A, then, according to Theorem 1 A is the extension of the quasicyclic subgroup by the finite Dedekind group, or A is the quasicentral subgroup of the group G.

Let A be the extension of the quasicyclic subgroup B by the finite Dedekind group. If B is an infinite subgroup of G containing A, then  $B \subseteq G$  and furthermore every quotient subgroup

 $B/A \subseteq G/A$ . Since G/A is the Dedekind group, A then satisfies the  $3^{rd}$  condition of this lemma.

If A is a quasicentral subgroup of group G, then, analogous to the paragraph above, we can prove that G/A is the Dedekind group. Admittedly, G/A is the abelian or the Hamiltonian group. If G/A is the abelian group, then A satisfies the 1<sup>st</sup> condition of this lemma.

Let G/A be the Hamiltonian group. By Proposition 2 G/A is a locally finite group. Thus an extension of a locally finite group by a locally finite group is a locally finite group, which implies that G is a locally finite group and hence A satisfies the  $2^{nd}$  condition of this lemma.  $\blacksquare$ .

#### Lemma 4.

If G is the locally graded IAN group with the infinite nodal subgroup A non-quasicentral of G, then A is the extension of a quasicyclic group by the Dedekind group.

**Proof.** Let G be the locally graded IAN group with the infinite nodal subgroup A non-quasicentral of G. According to Theorem 1 A is a periodic group, by Lemma 3 A is the extension of a quasicyclic group by the Dedekind group.

## Lemma 5.

If G is the group with a finite nodal subgroup A, then G/A is the abelian group, or the group.

**Proof.** If G/A is the abelian group, then this lemma is valid. Let G/A be a non-abelian group and B/A be an arbitrary infinite subgroup of G/A. There evidently exists  $B ext{ } e$ 

#### Theorem 2.

Let G be the locally graded IAN group with a nodal subgroup A. The nodal subgroup A of G is a non-quasicentral of G if and only if it satisfies one of the following conditions:

- 1. A is a finite non-quasicentral subgroup of G, the quotient group G/A is the INH group with the abelian commutator or G/A is the abelian group;
- 2. A is an almost quasicyclic group which contains the finite subgroups that are not normal in G,  $|A:A\cap G| \le \infty$ , and G/A is the Dedekind group.

**Proof.** Let G be the locally graded IAN group with the infinite nodal subgroup A. Admittedly, the subgroup A is a non-quasicentral of G. Referring to Lemma 2 A is a finite group, or A is an almost quasicyclic group.

If A is a finite group then, by Lemma 5, G/A is the abelian group or G/A is a solvable INH group. According to Proposition 1 the commutator of a solvable INH group is the abelian group. Thus the subgroup A satisfies the 1st condition of this theorem.

Let G/A be a solvable *INH* group. By the condition 3 of Lemma 3 G/A A is the Dedekind group. Based on this fact A is an almost quasicyclic group and by Definition 1 A is a non-quasicentral of G. Suppose there exists a finite subgroup and A is normal in G. Therefore G/A is the Dedekind group, A is an almost quasicyclic group, thus G' is a subgroup of that almost quasicyclic subgroup A. G' of G'. Let G' be a finite group and put A=G'. Hence A satisfies either condition 1 or condition 2 of this theorem.

If G' is an infinite group, then  $|A.G':G'| < \infty$ ,  $|A:AG'| < \infty$ , too. Admittedly, A satisfies the  $2^{nd}$  condition of this theorem.

Conversely. Suppose the nodal subgroup A satisfies either condition 1 or condition 2 of this theorem, then G is the IAN group with the non-quasicentral nodal subgroup A.

By Theorem 2 and the definition of IANA groups the next corollary follows.

## Corollary 2.

Let G be a locally graded IANA group with a nodal subgroup A. The nodal subgroup A of G is a non-quasicentral of G if and only if it satisfies one of the following conditions:

- 1. A is the finite abelian non-quasicentral subgroup of G, the quotient group G/A is the INH group with the abelian commutator or G/A is the abelian group,
- 2.  $A = Z(p^{\infty}) \times D$  where D is the finite abelian subgroup of G, A contains finite subgroups that are not normal in G, and G/A is the Dedekind group.

# Lemma 6.

Let G be the IAN group with the infinite non-quasicentral Dedekind nodal subgroup A. Then  $A=Z(p^\infty)\times D$  where D is a finite Dedekind group,  $p\mid |D|$ , and there exists the element  $a\subseteq A$  so that the subgroup a>0 is not normal a>0 subgroup of G. For a>0 is a>0 is not normal a>0 subgroup of a>0.

**Proof.** Let G be the IAN group with the infinite non-quasicentral Dedekind nodal subgroup A. By Lemma 1 A is an almost quasicyclic group. Pursuant to Proposition 2  $A=Z(p^\infty)\times D$  where D is a finite Dedekind group, p=2, and D'=4. We shall prove that A does not contain a normal p-group 4 of 4.

Obviously, if every cyclic subgroup of A is a normal subgroup of G, then A is a quasicentral subgroup of G. Thus there exists a cyclic subgroup  $\langle x \rangle$  of A that is not normal in G, which implies that the  $\langle a \rangle$  Sylow G- subgroup of the group  $\langle x \rangle$  is not normal in G. This verifies that G = G.

Let  $q \neq p$  and  $Z(p\infty) \cap \langle a \rangle = \langle e \rangle$ . Then  $Z(p\infty), \langle a \rangle \cong Z(p\infty) \times \langle a \rangle$  where  $Z(p\infty) \times \langle a \rangle$  is the normal subgroup of G. Evidently,  $\langle a \rangle$  is the Sylow q-subgroup normal in  $Z(p\infty) \times \langle a \rangle$  and  $\langle a \rangle$  is normal in G. This is a contradiction, thus g = p.

If  $A = Z(p\infty) \times D \ge Z(p\infty) \le a$ , then  $Z(p\infty) \le a \ge Z(p\infty) \le b > 2 \ge 0$ . Because the subgroup  $\le a \ge 0$  is the p-subgroup

normal in G and furthermore  $Z(p^{\infty}) \cap \langle a \rangle < \langle a \rangle$ ,  $|b| > 1, \langle b \rangle$  is a p-group, therefore  $p \mid |D|$ .

According to Lemma 6 and the Definition of IANA groups, the next corollary follows.

# Corollary 3.

Let G be the IANA group with the infinite non-quasicentral nodal subgroup A. Then  $A=Z(p^{\infty})\times D$ , where D is a finite group,  $p \mid |D|$ , the subgroup D contains an element a A so that the  $\langle a \rangle$ p - subgroup is not normal in G.

#### Theorem 3.

The group G is the locally graded IAN group with the infinite non-quasicentral nodal subgroup A of G if and only if a quotient group G/A is the Dedekind group,  $|A:A\cap G'| \le and$  the nodal subgroup A is of one of the types:

- 1.  $A=Z(p\infty)\times D$ , where D is the finite Dedekind group, p=2, D'=4e,  $p \mid D \mid$ , the subgroup A contains an element a such that <a> p-subgroup is not normal in G, and the quotient group  $A/Z(p\infty)$  is the quasicentral in  $G/Z(p\infty)$ ;
- 2.  $A = Z(p\infty) \setminus D$ , where D is the finite Dedekind subgroup, the group A does not contain the finite normal subgroup of G, and  $A/Z(p^{\infty})$  is the quasicentral in  $G/Z(p^{\infty})$ ;
- 3.  $A=(Z(p^{\infty}).B)\times D$ , where  $Z(p^{\infty}).B$  is the non-abelian Syllow p-subgroup of G, D is the infinite Dedekind group, p=2,  $D'= <e>, Z(p\infty) \le C(G)$ , the finite group B has a normal series:  $Z(p\infty)\cap B=B'=B_0 \leq B_1 \leq \ldots \leq B_r \leq \ldots \leq B_r, n\geq 1, B_1=B_{1,1} \leq b_1 \leq s_1$ for all  $i \ge 1$ ,  $|b_i| \ge 1$ , and  $A/\mathbb{Z}(p^{\infty})$  is the quasicentral in  $G/Z(p\infty)$ ;
- 4.  $A = (Z(p^{\infty}), B) \times Q_{g}) \times D$ , where  $Z(p^{\infty}), B, Q_{g}$  is the Sylow 2-subgroup of G, D is the finite abelian group,  $Z(p^{\infty}) \leq C(G)$ , the finite group B has a normal series:  $Z(p\infty)\cap B = \langle B', [B, Q_8] \rangle = B_0 \langle B_1 \langle ... \langle B_i \langle ... \langle B_n \rangle, n \geq 1,$

 $B_i=B_{i,1} \times \langle b_i \rangle$ , for all  $i \ge 1$ ,  $|b_i|=2$ ,  $|B_0|=2$ , and  $A/Z(p^\infty)$  is

the quasicentral in  $G/Z(p\infty)$ ;

- 5.  $A = (Z(2^{\infty}).B)$ .  $\langle d \rangle \times D$ , where  $Z(2^{\infty}).B. \langle d \rangle$  is the Sylow 2 - subgroup of G, D is the finite abelian group,  $Z(2^{\infty}) \le C(Z(2^{\infty}).B)$ , for each  $c \in Z(2^{\infty})$ ,  $d^1cd=c^1$ , the finite group B has a normal series :  $Z(2^{\infty}) \cap B = \langle B', [B, \langle d \rangle] \rangle = B_0 \langle B, \langle d \rangle$  $\langle B_i \langle ... \langle B_n, n \geq 1, B_i = B_{i,1} \rangle \langle d \rangle$ , for all  $i \geq 1$ ,  $(Z(p^{\infty}). B) \cap \langle d \rangle =$  $Z(2^{\infty}) \cap \langle d \rangle \leq \langle c_1 \rangle$ ,  $|c_1| = 2$ , and  $A/Z(p^{\infty})$  is the quasicentral in  $G/Z(p^{\infty})$ ;
- 6.  $A=(((Z(2^{\infty}).B).< d>).< d>). Q_g)\times D$ , where  $Z(2^{\infty}).B$ .  $Q_g.< d>$  is the Sylow 2- subgroup of G, D is the finite abelian group,  $Z(2^{\infty}) \leq C(Z(2^{\infty}).B. Q_{s})$ , for each  $c \in Z(2^{\infty})$ ,  $d^{-1}cd=c^{-1}$ , the finite group B has a normal series:
  - $\langle B_i \leq ... \leq B_{n'} \ n \geq 1, \ B_i = B_{i,1} \searrow \langle b_i \rangle, \ for \ all \ i \geq 1, \ | \ b_1 \ | \ = 2, \ | \ B_0 \ | \ = 2,$  $\mid d \mid = 4$ ,  $(Z(p\infty).B) \cap \langle d \rangle = Z(2^{\infty}) \cap \langle d \rangle$ , and  $A/Z(p\infty)$  is the *quasicentral in*  $G/Z(p\infty)$ ;
- 7.  $A=((((Z(2^{\infty}).B) \leftthreetimes Q_{s}) \times D.B) \leftthreetimes Q_{s}) \times D$ , where  $Z(2^{\infty}).B.$   $Q_{s}$  is the Sylow 2- subgroup of G, D is the finite abelian group,

 $Z(2^{\infty}) \leq C(Z(2^{\infty}).B)$ ,  $[Z(2^{\infty}), Q_{g}] = Z(2^{\infty})$ , the finite group B has a normal series:

 $Z(2^{\infty}).B = \langle B', [B, Q_8] \rangle = B_0 \langle B_1 \langle ... \langle B_i \langle ... \langle B_n \rangle, n \geq 1,$  $B_i=B_{i,1}$ )  $\leftthreetimes \langle b_i \gt$ , for all  $i \ge 1$ ,  $|b_i|=2$ ,  $|B_0|=2$  and  $A/Z(p^\infty)$  is the quasicentral in  $G/Z(p^{\infty})$ ;

 $A=((Z(2^{\infty}).B).H) \times D$ , where  $Z(2^{\infty}).B$ . H is the Sylow 2-subgroup of G, D is the finite abelian group,  $Z(2^{\infty}) \le$  $\leq C(Z(2^{\infty}).B), [Z(2^{\infty}).H] = Z(2^{\infty}), H = \langle a \rangle \times \langle b \rangle, |a| = |b| =$ = 4, [a,b]= $a^2$ , the finite group B has a normal series:

 $Z(2^{\infty}) \cap B = \langle B', [B,H] \rangle = B_0 \langle B_1 \langle ... \langle B_i \langle ... \langle B_n, n \geq 1, n \rangle \rangle$  $B_i = B_{i,1} \setminus \langle b_i \rangle$ , for all  $i \ge 1$ ,  $|b_1| = 2$ ,  $|B_0| = 2$ ,  $(Z(2^{\infty}).B) \cap H =$  $=Z(2^{\infty})\cap H=\langle a^2, b^2\rangle$ , and  $A/Z(p^{\infty})$  is a quasicentral in  $G/Z(p^{\infty})$ .

**Proof.** Let G be the locally graded IAN group with the infinite non-quasicentral nodal subgroup A of G. By Theorem 1 A is an almost quasicyclic group containing the finite subgroups that are not normal in G,  $|A:A\cap G| \le \infty$ , and G/A is the Dedekind group. The above mentioned implies that A contains a subgroup  $Z(p^{\infty})$ that is normal in G,  $A/Z(p^{\infty})$  is the finite Dedekind group and furthermore  $Z(p\infty) \le B \le A$ . Pursuant to Definition 1 the group B is the infinite subgroup of G. Admittedly, B is normal in Gand the factor group  $A/Z(p^{\infty})$  is the quasicentral subgroup of  $G/Z(p^{\infty})$ . According to Theorem 3.1 [8] the subgroup A satisfies the conditions of this theorem. Evidently, A is the group of one of types 1 to 8 of this theorem.

If A is a group of the type 1 of Theorem 3.1 [8], then A is the Dedekind group,  $A = Z(p^{\infty}) \times D$  where D is the finite Dedekind group, p = 2, and D' = <e>. By Lemma 6  $p \mid |D|$ , the subgroup A contains element a so that a subgroup  $\langle a \rangle$  is not normal p-subgroup in G. Thus A is of the type 1 of this theorem.

If A is a group of one of the types 2 - 8 of Theorem 3.1 [8], then A is a subgroup of one of the types 2-8 of this theorem.

Conversely. If G is a group with the normal subgroup A of one of the types 1 - 8 of this theorem, then G/A is the Dedekind group. G is evidently the locally graded group. Because G/A is the Dedekind group and  $A/Z(p^{\infty})$  is the quasicentral subgroup of  $G/\mathbb{Z}(p^{\infty})$ , then any infinite subgroup contained in A and any subgroup which contains a subgroup A is normal in G. Thus Gis the IAN group.

Let A be an infinite subgroup of G. If the subgroup A is of the type 1 of this theorem, then the subgroup A contains a subgroup  $\langle a \rangle$  that is not normal in G. Thus the subgroup A is nonquasicentral subgroup of G.

Thus the quasicentral subgroups of the group G are the Dedekind groups, which implies A is a group of one of the types 2 - 8 of this theorem. Thus A is the non-Dedekind group, which implies that the subgroup A of one of the types 2 - 8 is a nonquasicentral subgroup of G.

# COMMUNICATIONS

#### Theorem 4.

The group G is the IANA group with an infinite non-quasicentral nodal subgroup A, if and only if  $A=Z(p^\infty)\times D$ , where D is a finite group,  $p\mid D\mid$ , the subgroup A contains an element A so that A p-subgroup is not normal of A, and  $A/Z(p^\infty)$  is the quasicentral in A

**Proof.** Let G be the locally graded IAN group with the infinite non-quasicentral nodal subgroup A of G, and A' = <e>. Because G/A is the Dedekind group,  $A = Z(p) \times D$ , where D is the finite abelian group, A contains the finite subgroups that are not normal in G, and G/A is the Dedekind group. The group G is evidently the locally graded IAN group with the nodal subgroup A of the type 1 of Theorem 2,  $p \mid D$ , the subgroup A contains

an element a so that  $\langle a \rangle$  p - subgroup is not normal in G, and  $A/\mathbb{Z}(p^{\infty})$  is the quasicentral in  $G/\mathbb{Z}(p^{\infty})$ .

Let G be a group, A is a subgroup A of G, and  $A=Z(p^{\infty})\times D$ , where D is a finite group,  $p\mid |D|$ , the subgroup A contains an element a so that  $\langle a \rangle p$  - subgroup is not normal in G, and  $A/Z(p^{\infty})$  is the quasicentral in  $G/Z(p^{\infty})$ .

Conversely. Suppose that  $A \subseteq G$  where A is an almost quasicyclic group. Since  $A/Z(p^{\infty})$  is a quasicentral in  $G/Z(p^{\infty})$ , then  $B/Z(p^{\infty}) \subseteq G/Z(p^{\infty})$  for all  $B/Z(p^{\infty}) \subseteq A/Z(p^{\infty})$ . Hence

 $B \subseteq G$ , A is the abelian subgroup, every infinite subgroup of A and every infinite subgroup of G containing A is a normal subgroup of G. By Definition 1 the group G is the IANA group. Hence the subgroup A contains the subgroup that is not normal in G, then A is the non-quasicentral in G.

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