MONOMIAL CURVES IN AFFINE SPACE AND THEIR ASSOCIATED PRIME IDEALS WITH SIX GENERATORS AS SET-THEORETIC COMPLETE INTERSECTIONS

The paper deals with the problem of the expression of associated prime ideals of monomial curves in the affine space $A^n$ as set-theoretic complete intersections. We describe some associated prime ideals, a minimal generating set of which has six elements and we prove that these ideals are set-theoretic complete intersections. Corresponding monomial curves are intersections of three hypersurfaces and we find the equations of these hypersurfaces.

Keywords: A monomial curve, an associated prime ideal, a set-theoretic complete intersection.

1. Introduction

It is known that $k$-dimensional algebraic affine variety is intersection of not fewer than $n - k$ hypersurfaces in $n$-dimensional affine space $A^n$. There is the presumption, that a number of these hypersurfaces is exactly $n - k$. In this case we can say, that they are ideal-theoretic or set-theoretic complete intersections. This is also equivalent to the fact, that either the associated ideal $I$ of this variety has generators (ideal-theoretic complete intersection) or the ideal $I$ is radical of an ideal $a$, $a \subseteq I$, the ideal $a$ has $n - k$ generators (set-theoretic complete intersection). The number $n - k$ is also height of the ideal $I$. The ideal is called a set-theoretic complete intersection (s.t.c.i., for short), if there are $s = \text{ht}(I)$ elements $g_1, g_2, \ldots, g_s$ such that $\text{rad}(I) = \text{rad}(g_1, g_2, \ldots, g_s)$.

Let $K$ be an arbitrary field, $R = K[x_1, x_2, x_3, x_4]$ the polynomial ring in four variables over $K$. $C = C(n_1, n_2, n_3, n_4)$ a monomial curve in affine space $A^4$ over $K$ having parameterization $x_i = t^n, i \in \{1, 2, 3, 4\}$, where $n_1, n_2, n_3, n_4$ be positive integers with g.c.d. equal 1 and $n_1, n_2, n_3, n_4$ is a minimal set of generators for the numerical semigroup $H = \langle n_1, n_2, n_3, n_4 \rangle$.

The ideal $P$ of all polynomials $f(x_1, x_2, x_3, x_4) \in R$ such that $f(t^n, t^n, t^n, t^n) = 0$, $t$ transcendental over $K$, is the associated prime ideal of ring $R$ of the monomial curve $C$. $P$ is the corresponding ideal with dim$(P) = 1$ and height ht$(P) = 3$. In particular, the associated prime ideal $P$ of monomial curve $C$ in $A^4$ is a s.t.c.i. if $P = \text{rad}(g_1, g_2, g_3)$ and also a monomial curve $C$ is a s.t.c.i. (more information in [1]).

The general problem of whether all associated ideals of monomial curves (or monomial curves) are s.t.c.i. is still open. There are nevertheless some partial results in this direction. E. Kunz [1] showed that every monomial curve in 3-dimensional affine space is a s.t.c.i.

In 4-dimensional affine space $A^4$, H. Bresinsky proved that if numerical semigroup $H$ is symmetric, then the monomial curve $C(n_1, n_2, n_3, n_4)$ and its associated prime ideals are s.t.c.i. (see [2]). D. Patil presented in [3], if $n - 1$ numbers among $n_1, n_2, \ldots, n_4$ form an arithmetic sequence, then $C = C(n_1, n_2, n_3, n_4)$ in $A^4$ is s.t.c.i. S. Solcan dealt with monomial curves $C(p^1, p^2, p^3, p^4 + 1, (p + 1)^2)$ in $A^4$ as s.t.c.i. for the positive characteristic of the field $K$ char $K = p \neq 0$ [4] and for char $K = 0$ [5]. W. Gastinger in [6] proved that associated prime ideals of monomial curves in $A^4$ are s.t.c.i. if minimal generating sets of these ideals have four generators. We showed that associated prime ideals of monomial curves whose minimal set of generators have five elements is s.t.c.i. [7].

2. The associated prime ideal $P$ of the monomial curve $C$

Let a binomial term $\prod_{i=1}^{t_i} x_i^r - \prod_{i=1}^{t_i} x_i^s \in P$, where $\gamma' \phi_i = 0$, $i \in \{1, 2, 3, 4\}$. It is clear that $\sum_{i=1}^{t_i} \gamma' \phi_i = \sum_{i=1}^{t_i} \phi_i n_i$. We have basically two types of binomial terms of $P$: $x_i^r x_i^s - x_i^{r'} x_i^{s'}$ or $x_i^r x_i^s x_i^c - x_i^{r'} x_i^{s'} x_i^{c'}$. We denote the binomial term $x_i^r x_i^s x_i^c$ by $(x_i^r)$. If $r_i$ is minimal and by $(x_i^r, x_i^s)$ if $x_i^r x_i^s x_i^c \in P$ with $r_i$ minimal and $\alpha_j = r_i, \alpha_i = \alpha_j = 0$. Every generating set for

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Lemma 2.1 Let \( \prod_{i \in \{1, 2, 3, 4\}} x_{ij} \) be a monomial term in \( R \). Let \( (x', x') \) be a monomial curve in \( A^4 \). We can find in [6], Lemma 7.1, the next property of minimal generating set for a prime ideal \( P \).

We write \( x^\gamma x^\nu \neq x^\sigma x^\tau \) if either \( \gamma > \nu \) and \( \gamma > \sigma \) or the inequalities are reversed. For binomials we write \( x^\gamma x^\nu - x^\sigma x^\tau \neq x^\gamma x^\nu - x^\sigma x^\tau \) and \( x^\gamma x^\nu \neq x^\sigma x^\tau \). We next define a set \( D_{ij} \) gives

Corollary 2.1

1. Let \( D = \{ (x_i, x_j, (x_i', x_i') \} \) and \( x^\gamma x^\nu - x^\sigma x^\tau \in D_{ij} \), \( (i, j, k, l) = \{1, 2, 3, 4\} \). Then \( \gamma > \nu < r \), \( h \in \{1, 2, 3, 4\} \).

2. Let \( D = \{ (x_i', x_i'), (x_i, (x_i')) \} \). If \( x^\gamma x^\nu - x^\sigma x^\tau \in D_{ij} \), \( (i, j, k, l) = \{1, 2, 3, 4\} \), then \( \gamma < \nu < r \), \( h \in \{1, 2, 3, 4\} \).

3. Let \( D = \{ (x_i', x_i), (x_i', (x_i')) \} \). If \( x^\gamma x^\nu - x^\sigma x^\tau \in D_{ij} \), \( (i, j, k, l) = \{1, 2, 3, 4\} \), then \( \gamma < \nu < r \), \( h \in \{1, 2, 3, 4\} \).

4. Let \( D = \{ (x_i', x_i), (x_i', (x_i')) \} \subseteq C \subseteq \{ (x_i', (x_i')) \} \). If \( x^\gamma x^\nu - x^\sigma x^\tau \in D_{ij} \), \( (i, j, k, l) = \{1, 2, 3, 4\} \), then \( \gamma < \nu < r \), \( h \in \{1, 2, 3, 4\} \).

In [8], H. Bresinsky gives the following theorem.

Theorem 2.1

\( M = B \cup D_{ij} \cup D_{ij} \cup \{ (i, j, k, l) = \{1, 2, 3, 4\} \} \) is a minimal generating set for the associated prime ideal \( P = \mathcal{P}(n_1, n_2, n_i, n_1) \) of the monomial curve in \( A^4 \).

We can find in [6], Lemma 7.1, the next property of minimal generating set for a prime ideal \( P \).

Lemma 2.2 Let \( g_i = \prod_{i \in \{1, 2, 3, 4\}} x_i^j \) be a monomial term in \( R \). Let \( M = \{ (x_i') \} \) be a minimal generating set for the associated prime ideal \( P \) of the monomial curve in \( A^4 \). If \( x_i' \) and \( x_i^j \) have an inequality, then there is an integer \( k \), \( 3 < k < t \) with \( g_i \neq x_i^j \).

Theorem 2.2 Let \( B = \{ (x_i', (x_i'), (x_i', (x_i')) \} \subseteq M \), where \( M \) is a minimal generating set for the associated prime ideal \( P \) of a monomial curve in \( A^4 \). If \( x_i^j \) is a minimal generating set for the associated prime ideal \( P \) of a monomial curve in \( A^4 \).

Theorem 2.3 Let \( B = \{ (x_i', (x_i'), (x_i', (x_i')) \} \subseteq M \), where \( M \) is a minimal generating set for the associated prime ideal \( P \) of a monomial curve in \( A^4 \).

Theorem 2.4 If \( B = \{ (x_i', (x_i'), (x_i', (x_i')) \} \subseteq M \), \( (i, j, k, l) = \{1, 2, 3, 4\} \), and \( M \) is a minimal generating set for the associated prime ideal \( P \) of a monomial curve in \( A^4 \).

When we use permutation of index \( (i, j, k, l) \) in Theorem 2.5 we receive following.

Corollary 2.2 Let \( B = \{ (x_i', (x_i'), (x_i', (x_i')) \} \subseteq M \), \( (i, j, k, l) = \{1, 2, 3, 4\} \), and \( M \) is a minimal generating set for the associated prime ideal \( P \) of a monomial curve in \( A^4 \).

3. The minimal generating set for a prime ideal \( P \) having six generators

The induct propositions give necessary conditions on a minimal set of generators for an associated prime ideal \( P \) of a monomial curve in \( A^4 \). When we use methods presented in [9]...
[1] and we also suppose that the minimal generating set \( M \) has six generators:
\[
M = \{ x_i^a - x_j^b x_k^c x_t^d, x_i^e - x_j^f x_k^g x_t^h, x_i^a - x_j^b x_k^c x_t^d, x_i^e - x_j^f x_k^g x_t^h, x_i^a - x_j^b x_k^c x_t^d, x_i^e - x_j^f x_k^g x_t^h \}
\]
where \( \{i,j,k,l\} = \{1, 2, 3, 4\} \) and \( \alpha_a \) and \( \alpha_b \) are not equal to zero, we get that there is an \( \alpha \in \mathbb{N} \) and all exponents must satisfy the following equations:
\[
r_r = \alpha_a + \alpha_b + (u + 1) \alpha_u,
\]
(2) where \( \alpha_a \) and \( \alpha_b \) are not equal to zero and exponents satisfy equations (2) for some \( u \in \mathbb{N} \), then this prime ideal \( P \) (monomial curve \( C \)) is an intersection of three hypersurfaces and we give the equations of these hypersurfaces:

**Theorem 4.1** Let \( P \) be the associated prime ideal of the monomial curve \( C \) in \( A^n \).

If \( M = \{ x_i^a - x_j^b x_k^c x_t^d, x_i^e - x_j^f x_k^g x_t^h, x_i^a - x_j^b x_k^c x_t^d, x_i^e - x_j^f x_k^g x_t^h, x_i^a - x_j^b x_k^c x_t^d, x_i^e - x_j^f x_k^g x_t^h \} \) is a minimal generating set for the prime ideal \( P \), where \( \{i,j,k,l\} = \{1, 2, 3, 4\} \), \( \alpha_a \) and \( \alpha_b \) are not equal to zero and exponents satisfy equations (2) for some \( u \in \mathbb{N} \), then this prime ideal \( P \) (monomial curve \( C \)) is a set-theoretic complete intersection.

**Proof.** To prove our claim we need to show that \( P = \text{Rad}(g_1, g_2, g_3) \), \( g_1, g_2 \in \mathbb{P} \), \( s \in \{1, 2, 3\} \). We find expression of polynomials \( g_1, g_2, g_3 \).\( P \) is a Groebner basis for \( (g_1, g_2, g_3) \) with respect to the lexicographic order, taking \( x_1 x_2 x_3 \) as the leading term. [12], Chapter I, § 3, Exercise 4 each element \( f \in P \) is uniquely expressed \( f = a_1 \cdot 1 + \ldots + a_n \cdot x_i \ldots \), \( a_1, \ldots, a_n \in \mathbb{Z} \). Clearly, the module \( R(g_1, g_2) \) has a linearly independent basis \( \{1, \ldots, x_i \} \) over \( \mathbb{K}[x_1, x_2] \), thus is free module over \( \mathbb{K}[x_1, x_2] \) and its rank is \( r_P \). Therefore

\[
F_1 = \sum_{a}^{\infty} b_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2} + \sum_{a}^{\infty} b_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2} + \sum_{a}^{\infty} b_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2} + \sum_{a}^{\infty} b_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2} + \sum_{a}^{\infty} b_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2} + \sum_{a}^{\infty} b_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2}
\]
and

\[
F_2 \equiv x_1^{\alpha_1} x_2^{\alpha_2} + \sum_{a}^{\infty} b_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2} + \sum_{a}^{\infty} b_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2} + \sum_{a}^{\infty} b_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2} + \sum_{a}^{\infty} b_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2} + \sum_{a}^{\infty} b_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2}
\]

We know that \( R(g_1, g_2, g_3) \) is a module over \( \mathbb{K}[x_1, x_2] \) and \( \{g_1, g_2, g_3\} \) is a Grobner basis for \( \mathbb{K}[x_1, x_2] \) with respect to the lexicographic order, taking \( x_1 x_2 x_3 \) as the leading term. [12], Chapter I, § 3, Exercise 4 each element \( f \in P \) is uniquely expressed \( f = a_1 \cdot 1 + \ldots + a_n \cdot x_i \ldots \), \( a_1, \ldots, a_n \in \mathbb{Z} \). Clearly, the module \( R(g_1, g_2) \) has a linearly independent basis \( \{1, \ldots, x_i \} \) over \( \mathbb{K}[x_1, x_2] \), thus is free module over \( \mathbb{K}[x_1, x_2] \) and its rank is \( r_P \). Therefore

\[
F_1 \equiv x_1^{\alpha_1} x_2^{\alpha_2} + \sum_{a}^{\infty} b_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2} + \sum_{a}^{\infty} b_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2} + \sum_{a}^{\infty} b_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2} + \sum_{a}^{\infty} b_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2} + \sum_{a}^{\infty} b_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2}
\]
and

\[
F_2 \equiv x_1^{\alpha_1} x_2^{\alpha_2} + \sum_{a}^{\infty} b_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2} + \sum_{a}^{\infty} b_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2} + \sum_{a}^{\infty} b_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2} + \sum_{a}^{\infty} b_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2} + \sum_{a}^{\infty} b_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2}
\]
Example 4.1. We take the minimal generating set \( M_t \) for the associated prime ideals \( P_t \) of the monomial curves \( C_{(1,15,18,35)} \) and \( C_{(20,25,39,41)} \) which is given by Computer algebra system Macaulay created by D.Bayer and M.Stillman.

\[ M_1 = \{ x_i - x_2 x_3, x_2 - x_1 x_3 x_i, x_1 - x_1 x_2 x_i, x_2 - x_1 x_3 x_i, x_1 x_2 - x_1 x_3 x_i, x_1 x_3 - x_1 x_2 x_i, x_2 - x_1 x_3 x_i, x_1 x_3 - x_1 x_2 x_i, x_2 - x_1 x_3 x_i \} \]

\[ M_2 = \{ x_1 - x_1 x_2 x_i, x_2 - x_1 x_3 x_i, x_3 - x_1 x_3 x_i, x_2 - x_1 x_3 x_i \} \]

The ideals \( P_t, t \in \{1, 2\} \) generating by these sets belongs to the case of Theorem 4.1 for \((i,j,k,l) = (2,3,4,1), u = 1 \) and \((i,j,k,l) = (3,4,2,1), u = 0 \)

Hence ideals \( P_t, t \in \{1, 2\} \) is s.t.c.i. \( P_t = \text{Rad}(g_j, x_i - x_1 x_2 x_i - x_1 x_3 x_i) \), where

\[ g_j = x_i^{20} - 6x_i x_1 x_2 x_3 x_4 \]

\[ P_t = \text{Rad}(g_j, x_i - x_1 x_2 x_i - x_1 x_3 x_i - x_1 x_2 x_3 x_i) \]

\[ g_j = x_i^{20} - 16x_i x_1 x_2 x_3 x_4 + 120x_i x_1 x_2 x_3 x_4 - 560x_i x_1 x_2 x_3 x_4 + \sum_{k=1}^{15} (-1)^{k-1} \binom{1}{k} x_i^{13-k} (x_1 x_2 x_3 x_4)_{1-k} \]

References