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NON-LINEAR MAPPINGS AND THEIR APPLICATIONS

In this contribution we discuss selected geometrical mappings that, although uncommon in practice, are important and irreplaceable in some areas of engineering, architecture and art (scenography, embossing, panoramic cinematography).

Keywords: Linear mapping, cylindrical and conical perspective, distance radius.

1. Introduction

Attempting to capture the surrounding world and preserve it in unchanged form has been a focus of interest of humans since the beginning of time [1]. One of the basic problems of this endeavor has been the problem of transforming the three-dimensional world into a two-dimensional medium (paper, photography, or more recently also computer screen). To this end one uses a particular mathematical transformation called projection. It defines how particular points of the mapped 3D body (space) are mapped into a 2D surface (plane). There are many ways to perform this operation and there are consequently many known types of such mappings (projections).

A mapping is called *linear* if it maps each line onto a line, i.e. any three different collinear points into three distinct collinear points (points are collinear if they belong to one line).

A linear mapping in E_2 with a Cartesian coordinate system (O, x, y) can be expressed using the equations:

$$\begin{aligned}x' &= a_1x + a_2y + a_3 \\ y' &= b_1x + b_2y + b_3,\end{aligned}\tag{1}$$

where the coordinates of a point A are $[x, y]$ and the coordinates of a point A' are $[x', y']$. The point A' is the image of the point A .

Mappings in the three-dimensional space are defined analogously to those in the plane. A *linear mapping* in E_3 with Cartesian coordinates (O, x, y, z) is defined as follows:

$$\begin{aligned}x' &= a_1x + a_2y + a_3z + a_4 \\ y' &= b_1x + b_2y + b_3z + b_4 \\ z' &= c_1x + c_2y + c_3z + c_4\end{aligned}\tag{2}$$

with point $A = [x; y; z]$ and its image point $A' = [x'; y'; z']$.

Nonlinear mappings are mapping that do not preserve lines, and thus deform their image. Such mapping can be defined by equations:

$$\begin{aligned}x' &= f(x, y, z) \\ y' &= g(x, y, z) \\ z' &= h(x, y, z)\end{aligned}\tag{3}$$

where $f(x, y, z)$, $g(x, y, z)$, $h(x, y, z)$, are nonlinear functions with variables x , y and z .

2. Applications of nonlinear mappings

Nonlinear mappings have applications in various areas such as in computer graphics where they, for instance, form the underlying basis of visual deformations and graphical special effects (Fig. 1).



Fig. 1 Deformed image

Cylindrical and conical perspectives are examples of nonlinear mappings. These mappings are the basis of cartographic projections onto cylindrical and conical surfaces which have already been in use since the ancient Greece, to construct maps of the world and the stars. For illustration, an early example of a cylindrical mapping is the so-called Marinus projection – from around the first century BC (Fig. 2) and an example of an early conical mapping is the Ptolemaic projection – from around 150 BC (Fig. 3) [2].

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Fig. 2 Marinus projection currently



Fig. 3 Ptolemaic projection currently

As far as applications in engineering and technology, we find cylindrical perspective in panoramic cinemas where the image is projected onto a wide screen curved into the shape of a cylindrical surface.

In art, especially in the 19th century, painters often captured significant historical events in panoramic paintings (of which only 33 are known to have survived until this day). Such event was, for example, the battle of Lipany. The author of this panoramic painting is Luděk Marold (1865-1898). Another painting using a cylindrical perspective is the Battle of Borodino (Fig. 4) by the

Russian painter Franz Alekseevitch Roubaud (1856-1928) who created it in 1912. The painting is 15 meters high, and when unwound, it is 115 meters wide. It is located in a panoramic museum in central Moscow [1].

3. Fundamentals of cylindrical perspective

In a cylindrical perspective an image is projected onto a cylindrical surface. Ordinarily it is more common to project onto a plane. So why introduce something like a cylindrical perspective and invent a complicated process of projecting an object on a cylinder and then transfer it onto the plane? The reason is simple – using a cylindrical perspective we can cover up to 360 degrees viewing angle. This cannot be achieved with a planar projection. Another advantage is that if we want to capture a very large object, for example, a street or a cityscape, we must either be at a great distance, thus losing precision and detail, or be closer but not able to capture everything we want. Cylindrical perspective allows us to map a wide object from a much smaller distance than what would be needed using other methods. Some detail is lost, but not as much as if we were far away from the object. So how does a cylindrical perspective work?

Let us have a cylindrical surface Φ with axis of rotation o , radius r (the distance radius) and a point S , which lies on the o axis. Let $\overline{E_3}$ denote E_3 augmented with points at infinity, and let G be a rotational conical area bounded by conical surface Ψ , axis-aligned with the surface Φ , with the point S as its peak point and apex angle of 90° . Under cylindrical perspective the image of a point $A \in \overline{E_3} - G$ is the intersection of the ray $\overline{SA}$ with the cylindrical surface Φ , i.e. $A_s = \overline{SA} \cap \Phi$ (Fig. 5).

Cylindrical perspective of objects is in descriptive geometry constructed by unwinding the cylindrical surface Φ . The



Fig. 4 F. A. Roubaud: Battle of Borodino

intersection of the surface Φ with the plane through the center S and orthogonal to the o axis is the circle h (horizon). By unwinding the horizon we obtain the line segment h_0 , whose

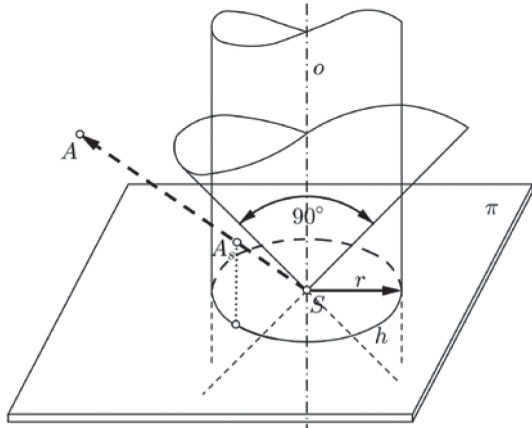


Fig. 5 The projection of the point

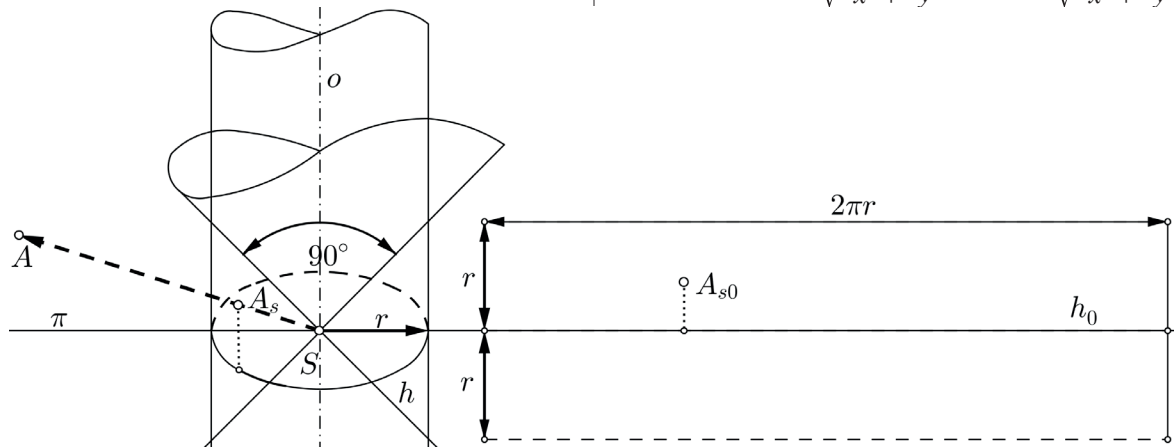


Fig. 6 Image point after unwinding area

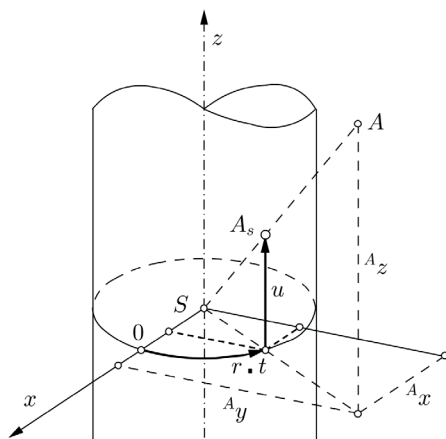


Fig. 7 Cartesian and cylindrical coordinates

length is $2\pi r$. The images of points in the unwound cylindrical perspective are located within the rectangle whose middle segment is h_0 and whose height is $2r$ (Fig. 6).

Consider a point $A = [{}^A x; {}^A y; {}^A z]$ in the Cartesian coordinate system (S, x, y, z) . In the plane of the unwound cylindrical surface Φ consider a Cartesian coordinate system (O, x', y') where the point $O = h \cap x$ and $x' \equiv h_0$.

Let A_s be the image of the point A in the cylindrical perspective, and let ${}^A x', {}^A y'$ be its coordinates in the unwinding of the cylindrical surface Φ .

The coordinate ${}^A x'$ of the point A_{s0} is equal to the length of the arc of the horizontal circle h bounded by the point O and the intersection of the horizon with the vertical (generating) line of the cylindrical surface passing through the point A_s . If t is the magnitude (in radians) of the angle spanned by this arc, then the coordinate satisfies ${}^A x' = r \cdot t$. The coordinate ${}^A y'$ is equal to the distance of the point A_s from the aforementioned intersection. Consequently the mapping equations of the cylindrical perspective can be formulated as follows:

$${}^A x' = r \cdot \arcsin \frac{{}^A y}{\sqrt{{}^A x^2 + {}^A y^2}}, \quad {}^A y' = r \frac{{}^A z}{\sqrt{{}^A x^2 + {}^A y^2}}.$$

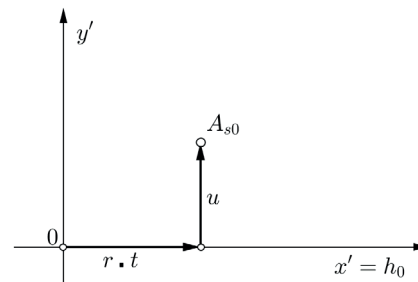


Fig. 8 Coordinates in the plane unwinding

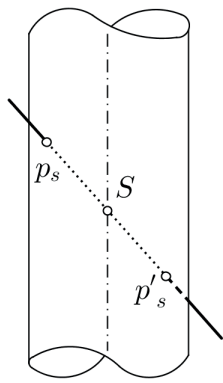


Fig. 9 $S \in p$

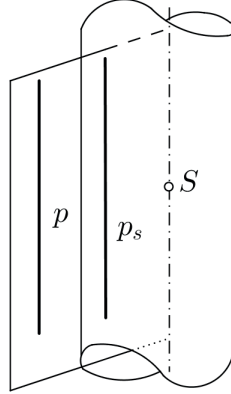


Fig. 10 $p \parallel o$

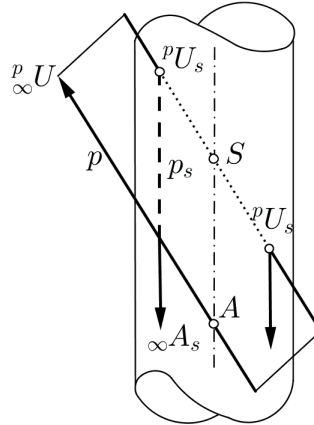


Fig. 11 $p \cap o = \{A\}$

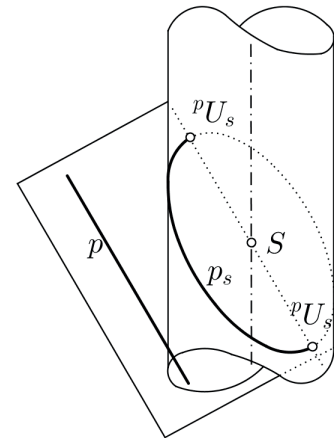


Fig. 12 Skew lines o, p

Let (S, x, y, z) be a given Cartesian coordinate system and let $A = [{}^4x; {}^4y; {}^4z]$ be a given point. Let Φ be a given cylindrical surface. Since $A_s = \vec{S}A \cap \Phi$, we can derive, from the Cartesian coordinates of the point A_s , the cylindrical coordinates: $x = r \cdot \cos t$, $y = r \cdot \sin t$, $z = u$ (Fig. 7). After unwinding the cylindrical surface Φ into a plane, in this plane we can define a new Cartesian coordinate system (O, x', y') in which ${}^4x' = r \cdot t$ and ${}^4y' = u$ (Fig. 8).

To find an image of a line in the cylindrical perspective it is important to determine what its relative position with respect to the center of projection S is and also with respect to the o axis of the cylindrical surface Φ .

Theorem 1: For a cylindrical perspective of a line p ($p \neq o$) the following holds [3]:

- If $S \in p$, then the image of p are two points p_s and p'_s , namely the points of intersection of the line p with the cylindrical surface Φ (Fig. 9).
- If p and o are parallel, then the image of p is the generating line p_s of the cylindrical surface Φ (Fig. 10).
- If p and o are intersecting, then the image of p is formed by two rays located at generating lines of the cylindrical surface Φ (Fig. 11).
- If none of the conditions a) - c) holds, then the image of p is a semi-ellipse or a semi-circle (Fig. 12).

Theorem 2: Unwinding an ellipse lying on a cylindrical surface Φ yields a part of the sinusoid [4].

Proof: The parametric equations of points on a rotational cylindrical surface Φ around the axis z with radius r are as follows: $x = r \cdot \cos t$, $y = r \cdot \sin t$, $z = u$, $t \in \langle 0, 2\pi \rangle$, $u \in R$. Consider

a plane α containing the x -axis forming an angle φ ($\tan \varphi = \frac{b}{r}$) with the plane (x, y) (Fig. 13). Then the analytical expression of

the points on the plane α is: $z = \frac{b}{r} \cdot y$. Substituting $y = r \cdot \sin t$,

we obtain the expression for the z -coordinates of the points of the cut k (the intersection of Φ and α):

$$z = \frac{b}{r} \cdot y = \frac{b}{r} \cdot r \cdot \sin t = b \cdot \sin t.$$

If $b \neq 0$, then the intersection of Φ with the plane α is the ellipse k . The coordinates of the points of the ellipse after unwinding the surface are: $x' = r \cdot t$, $y' = z = b \cdot \sin t$, thus the unwound ellipse forms a part of a sinusoid which can be expressed in the form of the function: $y' = b \cdot \sin \frac{x'}{r}$.

Theorem 3: Let $\alpha \cap \alpha' = p$, where p is the diameter of the circle h . Then unwinding the ellipses k and k' produces sinusoids where one is an affine projection of the other, with the axis of the affinity being h_0 and the direction perpendicular to the axis.

Fig. 13 $\alpha \cap \Phi = k$

Proof: Using the proof of Theorem 2, let $\alpha \cap \alpha' = x$ and let the angle of the plane α' with $\pi = (x, y)$ be ϕ' , i.e. $\tan \phi' = \frac{b'}{r}$ (Fig. 14). For the coordinates of the points on the ellipse k' after unwinding the surface, we have: $x' = r \cdot t$ and $y' = z = b' \cdot \sin t$. From both expressions for y' , it follows that the affinity with axis $x' \equiv h_0$, with the direction perpendicular to the axis, and with characteristic $\frac{b'}{b}$, maps the sinusoid k_0 to the sinusoid k'_0 (Fig. 15).

When unwinding the curve that is cut on the surface by the plane, we assume that its starting point either lies on the positive part of the x -axis, or lies on a line parallel to the x -axis with a positive x -coordinate.

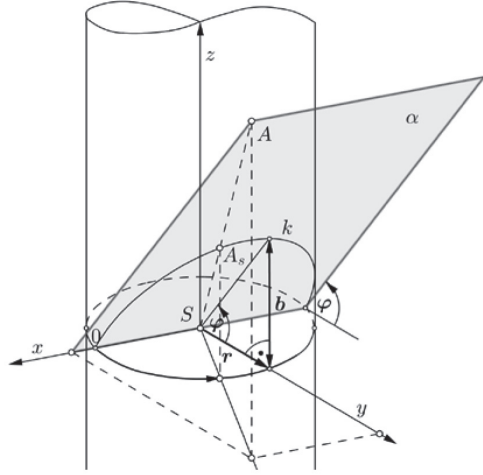


Fig. 13 $a \cap \phi = k$

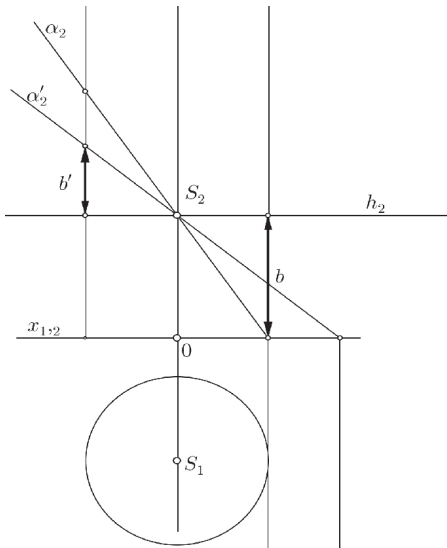


Fig. 14 Cuts on the cylinder

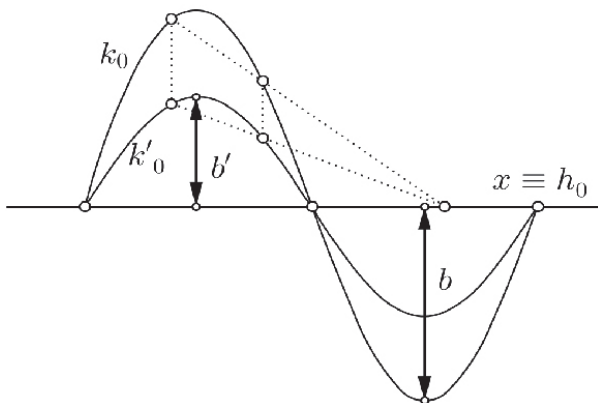


Fig. 15 Affine relationship of ellipses

4. Fundamentals of conical perspective

In this section, we discuss the conical perspective, i.e. a mapping onto a rotational conical surface Ψ from a point S (center of projection) of its axis (the point S is distinct from the peak point V of this conical surface). The field of view of the conical perspective, i.e. a restricted part P'_3 of the space $P_3 = \overline{E}_3$, can be determined similarly as for the cylindrical perspective.

Consider a rotational conical surface Ψ with axis o and point S which lies on the o axis.

Then, a conical perspective of a point $A \in P'_3$ is the point of the intersection of the ray $\overrightarrow{SA}$ with the conical surface Ψ , i.e. $A_s = \overrightarrow{SA} \cap \Psi$ (Fig. 16).

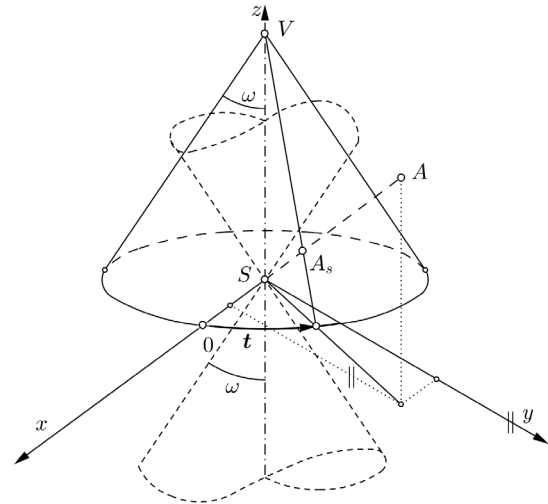


Fig. 16 Conical perspective of a point

If there are two intersection points of the ray $\overrightarrow{SA}$ with the conical surface Ψ , the conical perspective of A is defined to be the point that is closer to the point S [4].

Let us fix a coordinate system (S, x, y, z) in P'_3 and consider a conical surface Ψ with axis z , where ω is the angle of the generating line with the axis. In a conical perspective with center S onto the surface Ψ with radius r of the horizon, the image of $A_s = [^A x; ^A y; ^A z]$ of a point $A = [^A x; ^A y; ^A z]$ satisfies the following equations:

$$\begin{aligned} ^{A_s} x &= \frac{r \ ^A x^2}{^A y^A z \tan \omega + ^A x \sqrt{^A x^2 + ^A y^2}}, \\ ^{A_s} y &= \frac{r \ ^A y^2}{^A y^A z \tan \omega + ^A x \sqrt{^A x^2 + ^A y^2}}, \\ ^{A_s} z &= \frac{r \ ^A y^A z}{^A y^A z \tan \omega + ^A x \sqrt{^A x^2 + ^A y^2}}. \end{aligned}$$

The first step in the construction of the image of a point in the conical perspective is to construct the point $A_s = \overrightarrow{SA} \cap \Psi$ (Fig. 17) which we usually perform using orthographic (Monge) projection. The next step is to unwind the conical

surface Ψ into the plane. This allows us to obtain the angle $\alpha^\circ = 360^\circ \cdot \sin \omega$, or $\alpha = 2\pi \cdot \sin \omega$ (in radians) where the angle of generating lines with the axis is ω .

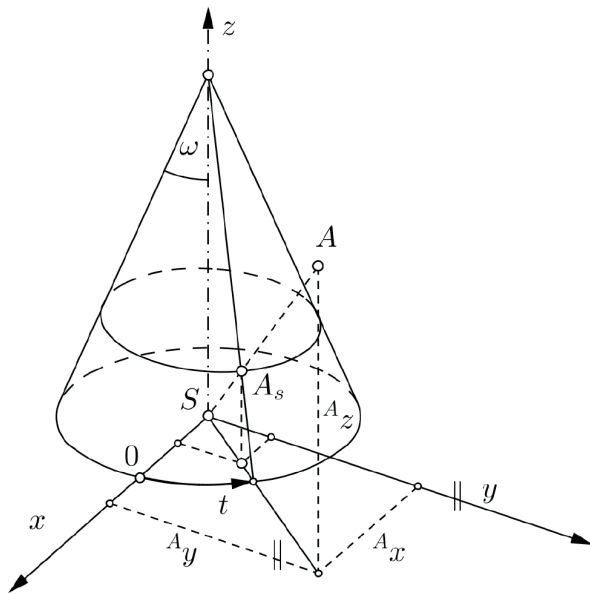


Fig. 17 Construction of image point

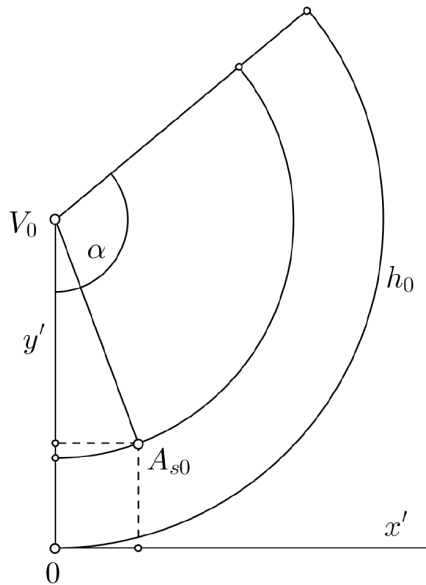


Fig. 18 Unwound surface

Generating lines of the conical surface are unwound into a pencil of lines with center V_0 and parallel circles map onto arcs of concentric circles whose center is again V_0 (Fig. 18).

Let A_{s0} be the location of the point A_s after unwinding the surface Ψ where $A_{s0} = [x'; y']$ are the coordinates of A_{s0} in the Cartesian coordinate system (O, x', y') . The coordinate x' is equal

to the unwound position of the generating line of the conical surface that lies in the plane (x, z) (Fig. 18). Then the mapping equations of the unwound position are as follows:

$$x' = \frac{r\sqrt{{}^Ax^4(\operatorname{tg}^2\omega + 1) + {}^Ax^2({}^Ax^2 + {}^Ay^2\operatorname{tg}\omega)}}{\operatorname{tg}\omega({}^Ay^4\operatorname{tg}\omega + {}^Ax\sqrt{{}^Ax^2 + {}^Ay^2})}$$

$$\sin\left[\arcsin\left(\frac{{}^Ay}{\sqrt{{}^Ax^2 + {}^Ay^2}}\right)\sin\omega\right],$$

$$y' = \frac{r}{\sin\omega} - \frac{r\sqrt{{}^Ax^4(\operatorname{tg}^2\omega + 1) + {}^Ax^2({}^Ax^2 + {}^Ay^2\operatorname{tg}\omega)}}{\operatorname{tg}\omega({}^Ay^4\operatorname{tg}\omega + {}^Ax\sqrt{{}^Ax^2 + {}^Ay^2})}$$

$$\cos\left[\arccos\left(\frac{{}^Ay}{\sqrt{{}^Ax^2 + {}^Ay^2}}\right)\sin\omega\right].$$

For the image of a line in a conical perspective (just like with cylindrical perspective) it is important to determine what its position with respect to the center of projection S is and also with respect to the o axis of the conical surface Ψ .

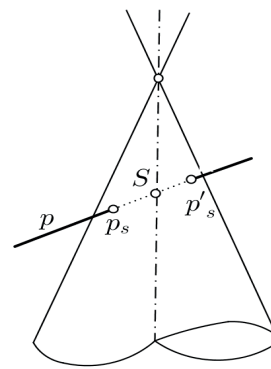


Fig. 19 $S \in p$

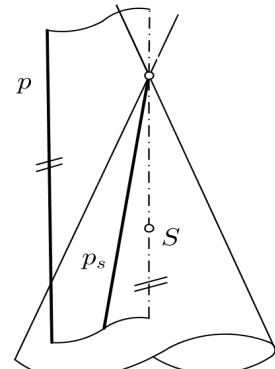


Fig. 20 $p \parallel o$

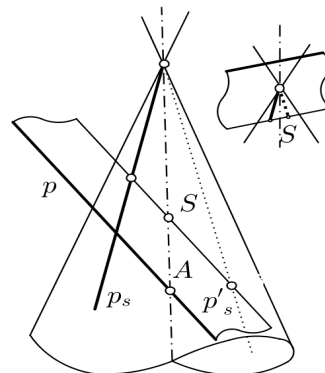


Fig. 21 $p \cap o = \{A\}$

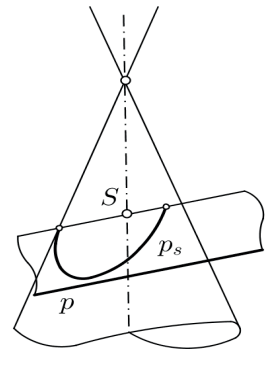


Fig. 22 Skew lines o, p

Theorem 4: If we consider also lines which are located outside the field of view, then for a conical perspective of a line p ($p \neq o$) the following holds [3]:

- If $S \in p$, then its image are two points p_s, p'_s , namely the points of intersection of the line p with the conical surface Ψ (Fig. 19).

- b) If p is parallel to o , then its image is p_s – a part of a generating line of the conical surface Ψ starting at the peak point (Fig. 20).
- c) If p intersects o , then its image is formed by two rays (or segments) lying on generating lines of the conical surface Ψ (Fig. 21).
- d) If none of the conditions a) – c) holds, then the image of p is part of a conic section (Fig. 22).

An image of a conic section lying in one plane with the x axis is, after unwinding the conic surface, a curve with parametric equations as follows:

$$x = \cos(t \sin \omega) \sqrt{\left[\frac{r \cos t}{1 + \operatorname{tg} \omega \operatorname{tg} \phi \sin t} \right]^2 + \left[\frac{r \sin t}{1 + \operatorname{tg} \omega \operatorname{tg} \phi \sin t} \right]^2} + \left[\frac{r}{\operatorname{tg} \omega} - \frac{r \operatorname{tg} \phi \sin t}{1 + \operatorname{tg} \omega \operatorname{tg} \phi \sin t} \right]$$

$$y = \sin(t \sin \omega) \sqrt{\left[\frac{r \cos t}{1 + \operatorname{tg} \omega \operatorname{tg} \phi \sin t} \right]^2 + \left[\frac{r \sin t}{1 + \operatorname{tg} \omega \operatorname{tg} \phi \sin t} \right]^2} + \left[\frac{r}{\operatorname{tg} \omega} - \frac{r \operatorname{tg} \phi \sin t}{1 + \operatorname{tg} \omega \operatorname{tg} \phi \sin t} \right]$$

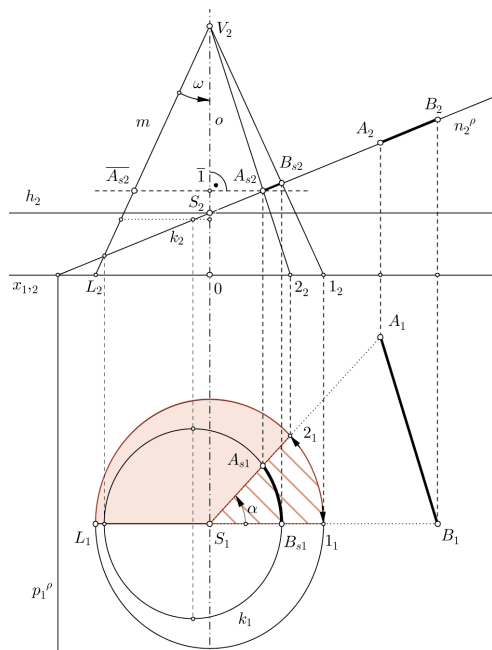


Fig. 23 The segment before unwinding the surface

where ω is the angle of generating lines of the conical surface Ψ with axis of rotation z , and ϕ is the angle of the projection plane with the plane of the horizon; t is a parameter.

5. Instead of final words

Exercise: Construct an image of the segment AB in conical perspective, if $SAB \perp v$ and $o \perp \pi$.

Construction: The first step is to construct a design cut k_1 conical surface Ψ of plane SAB ($A_{s1}, B_{s1} \in k_1$) in Monge's projection (Fig. 23). The next step is to unwinding the conical surface Ψ together with a cut k and with the points A_s, B_s into the plane (Fig. 24).

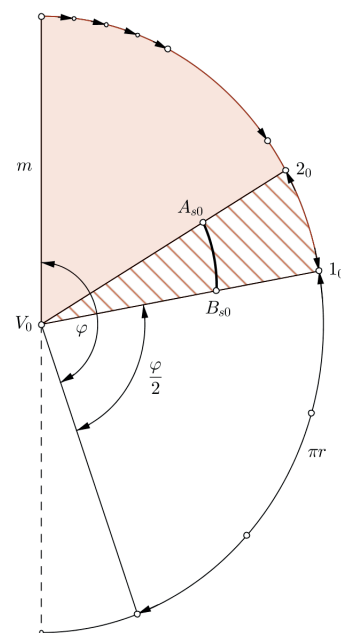


Fig. 24 Segment after unwinding

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