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TWO AND THREE-REVOLUTION CYCLICAL SURFACES

The creation of two-revolution and three-revolution cyclical surfaces is presented in the paper. Classification and vector equations of the surfaces are given. The surfaces are created by translation of the circle along the curves and its centre is on the curve. The curves are created by revolution of a point about any edge of the trihedron of the previous curve and this trihedron moves simultaneously along this curve. All specific forms of surfaces are illustrated in figures visualized in Maple.

1. Introduction

The point S_1 revolves about the coordinate axis z with angular velocity w_1 in the distance d_1 from the origin of the coordinate system (O, x, y, z). For every value of the angle w_1 there exists only one position of the point R_1 and the trajectory of this point R_1 is the curve k_1 (circle). The trihedron (R_1, t_1, n_1, b_1) defined in every point $R_1 \in k_1$ is determined by tangent, principal normal and binormal of the curve k_1. The point S_2 revolves at an angular velocity w_2 about any axis of the coordinate system, which is identical with the trihedron (R_1, t_1, n_1, b_1) of the curve k_1, in the distance d_2 from the origin of this coordinate system which is moving simultaneously along the curve k_1. For every value of the angle w_2 there exists only one position of the point R_2. The trajectory of this point R_2 is the curve k_2^g, where $g = t, n, b$. The trihedron (R_2, t_2, n_2, b_2) in every point $R_2 \in k_2^g$ is determined by the tangent, principal normal and binormal of the curve k_2^g. The point S_3 revolves about any axis of the coordinate system identical with the trihedron (R_2, t_2, n_2, b_2) of the curve k_2^g at an angular velocity w_3 in the distance d_3 from the origin of this coordinate system which is moving simultaneously along the curve k_2^g. For every value of the angle w_3 there exists only one position of the point R_3. The trajectory of the point R_3 is the curve k_3^{gh}, where $g, h = t, n, b$. The trihedron (R_3, t_3, n_3, b_3) in every point $R_3 \in k_3^{gh}$ is determined by the tangent, principal normal and binormal of the curve k_3^{gh}.

The surface of the type $P_1(u,v)$ is created by translation of the circle $c_1 = (R_1, r_1)$ along the curve k_1, the surface of the type $P_2^g(u,v)$ is created by translation of the circle $c_2 = (R_2, r_2)$ along the curve k_2^g and the surface of the type $P_3^{gh}(u,v)$ is created by translation of the circle $c_3 = (R_3, r_3)$ along the curve k_3^{gh}. The index $g = t, n, b$ determines that the point S_2 revolves about the tangent t_1, or principal normal n_1 or binormal b_1 of the curve k_1 and the index $h = t, n, g$ determines that the point S_3 revolves about tangent t_2, principal normal n_2 or binormal b_2 of the curve k_2^g.

2. Vector functions of the curves k_1, k_2^g, k_3^{gh}

Let the curve k_1 be a circle created by revolution of the point $S_1 = S_1(d_1, 0, 0, 1)$ about the axis z of the coordinate system (O, x, y, z) at an angular velocity $w_1 = v$ and k_1 is determined by the vector function

$$r_1(v) = (x_{k1}(v), y_{k1}(v), z_{k1}(v), 1) = S_1 \,.\, T_{z1}(w_1) =$$

$$= (d_1 \cos v, s_1 q_1 \sin v, 0,1),\ v \in \langle 0, 2\pi \rangle. \tag{1}$$

The matrix $T_{z1}(w_1)$ represents the revolution of the point S_1 about the coordinate axis z given by (5) (3rd matrix for $i = 1$), where the parameter $q_1 = \pm 1$ determines the right-turned or left-turned revolution movement of the point (Fig. 1, $i = 1, j = z$) [3]. We will define the trihedron (R_1, t_1, n_1, b_1) of the curve k_1 in every point $R_1 \in k_1$ by the tangent t_1, principal normal n_1 and by binormal b_1 with their unit vectors $t_1(v)$, $n_1(v)$, $b_1(v)$ by equations (2), (3), (4) for $i = 1$, $v \in \langle 0, 2\pi \rangle$

$$t_i(v) = (a_{ti}, b_{ti}, c_{ti}) = \frac{1}{h_{1i}} \frac{dr_i}{dv} = \frac{1}{h_{1i}} \left(\frac{dx_{ki}(v)}{dv}, \frac{dy_{ki}(v)}{dv}, \frac{dz_{ki}(v)}{dv} \right),\ h_{1i} = \left| \frac{dr_i}{dv} \right| \tag{2}$$

$$n_i(v) = (a_{ni}, b_{ni}, c_{ni}) = \frac{1}{h_{2i}} \frac{d^2 r_i}{dv^2} = \frac{1}{h_{2i}} \left(\frac{d^2 x_{ki}(v)}{dv^2}, \frac{d^2 y_{ki}(v)}{dv^2}, \frac{d^2 z_{ki}(v)}{dv^2} \right),\ h_{2i} = \left| \frac{d^2 r_i}{dv^2} \right| \tag{3}$$

$$b_i = \frac{1}{h_{3i}} (t_i(v) \times n_i(v)),\ h_{3i} = |t_i(v) \times n_i(v)|. \tag{4}$$

The curve k_2^g is created by revolution of the point S_2 in the distance d_2 from the origin of the coordinate system (O, x, y, z) about any coordinate axis x, y, or z through the angle w_2 into the point S'_2 (Fig. 1, $i = 2$ for , $j = x, y, z$). Angular velocity $w_2 = m_1 v$

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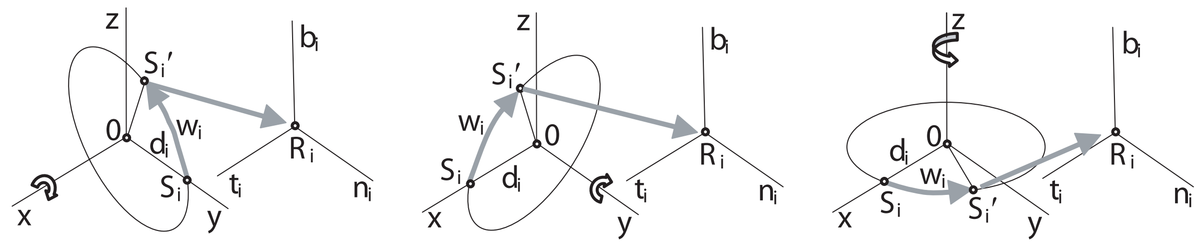

Fig. 1 Revolution of the point about axes x, y, z

of the point S_2 is m_1-multiple of angular velocity $w_1 = v$ of the point S_1 The point S_2' is transformed into the point R_2 in the coordinate system (R_1, t_1, n_1, b_1). If we create a surface of the type $P_2^g(u,v)$, where $g = t$ (or $g = n$, or $g = b$), we will revolve the point S_2 about the axis $j = x$, or $j = y$, or $j = z$. The revolution of the point S_2 is represented by a matrix $T_{j2}(w_2)$. $j = x, y, z$ in (5), where the parameter $q_2 = \pm 1$ determines the right-turned or left-turned revolution and the transformation of the point S_2' into the point R_2 is represented by a matrix $M_2(w_2)$ given by (5) [4]. The point S_2. will be situated always on any coordinate axis x, y, z.

$$T_{xi}(w_i) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos w_i & q_i \sin w_i & 0 \\ 0 & -q_i \sin w_i & \cos w_i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$T_{zi}(w_i) = \begin{pmatrix} \cos w_i & 0 & q_i \sin w_i & 0 \\ 0 & 1 & 0 & 0 \\ -q_i \sin w_i & 0 & \cos w_i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$T_{zi}(w_i) = \begin{pmatrix} \cos w_i & q_i \sin w_i & 0 & 0 \\ -q_i \sin w_i & \cos w_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5}$$

$$M_i(w_i) = \begin{pmatrix} a_{ti} & b_{ti} & c_{ti} & 0 \\ a_{ni} & b_{ni} & c_{ni} & 0 \\ a_{bi} & b_{bi} & c_{bi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The elements of the matrix $M_1(w_1)$ in (5) are the coordinates of unit vectors $t_i(v)$, $n_i(v)$, $b_i(v)$ of tangent t_i, principal normal n_i and binormal b_i in trihedron (R_i, t_i, n_i, b_i), $i = 1, 2, 3$. Then the vector function of the curve k_2^g is

$$r_2(v) = (x_{k2}(v), y_{k2}(v), z_{k2}(v), 1) = S_1 \,.\, T_{z1}(w_1) +$$

$$+ S_2 \,.\, T_{j2}(w_2) \,.\, M_2(w_2), \; j = x, y, z. \tag{6}$$

and $S_2 = S_2(d_2, 0, 0, 1)$ or $S_2 = S_2(0, 0, d_2, 1)$.

The trihedron (R_2, t_2, n_2, b_2) is determined in every point $R_2 \in k_2^g$ by the tangent t_2, principal normal n_2 and binormal b_2 with the unit vectors $t_2(v)$, $n_2(v)$, $b_2(v)$ expressed by equations (2), (3), (4) for $i = 2$.

The curve k_3^{gh} is created by revolution of the point R_3 in the distance d_3 from the origin of the coordinate system (O, x, y, z) about any coordinate axis x, y, or z through the angle w_3 into the point S_3' (Fig. 2, $i = 3$ for, $j = x, y, z$). Angular velocity $w_3 = m_2 w_2 = m_2 m_1 v$ of the point S_3 is m_2-multiple of angular velocity $w_2 = m_1 v$ of the point S_2. The point S_3' is transformed into the point R_3 in the coordinate system (R_2, t_2, n_2, b_2). If we create the surface of type $P_3^{gh}(u,v)$, where $h = t$ (or $h = n$, or $h = b$), we will revolve the point S_3 about the axis $j = x$, or $j = y$, or $j = z$. The revolution of the point S_3 is represented by a matrix $T_{j3}(w_3)$, $j = x, y, z$, where the parameter $q_3 = \pm 1$ determines the right-turned or left-turned revolution and transformation of the point S_3' into the point R_3 is represented by the matrix $M_3(w_3)$ by equations (5) [4]. The point S_3.will always be situated on any coordinate axis x, y, or z.

Then the vector function of the curve k_3^{gh} for $j = x, y, z$ is

$$r_3(v) = (x_{k3}(v), y_{k3}(v), z_{k3}(v), 1) = S_1 \,.\, T_{z1}(w_1) +$$

$$+ S_2 \,.\, T_{j2}(w_2) \,.\, M_2(w_2) + S_3 \,.\, T_{j3}(w_3) \,.\, M_3(w_3) \,,$$

$$j = x, y, z. \tag{7}$$

The trihedron (R_3, t_3, n_3, b_3) in every point $R_3 \in k_3^{gh}$ is determined by the tangent t_3, principal normal n_3 and binormal b_3 with the unit vectors $t_3(v)$, $n_3(v)$, $b_3(v)$ by equations (2), (3), (4) for $i = 3$.

In Fig. 2 there is displayed a revolution of the point S_1 about the coordinate axis z through the angle w_1 into the point R_1, where its revolutionary movement creates the curve k_1, revolution of the point S_2 about the coordinate axis z through the angle w_2 into the point S_2' and its transformation into the point R_2, where its revolutionary movement creates the curve k_2^g, revolution of the point S_3 about the coordinate axis z through the angle w_3 into the point S_3' and its transformation into the point R_3, where its revolutionary movement creates the curve k_3^{gh}. In the points R_1, R_2, R_3 there are displayed trihedrons (R_1, t_1, n_1, b_1), (R_2, t_2, n_2, b_2), (R_3, t_3, n_3, b_3).

In Fig. 3 there are displayed for illustration only three combinations of the curves k_1, k_2^t, k_3^{tt}, k_1, k_2^n, k_3^{nn} and k_1, k_2^b, k_3^{bb}.

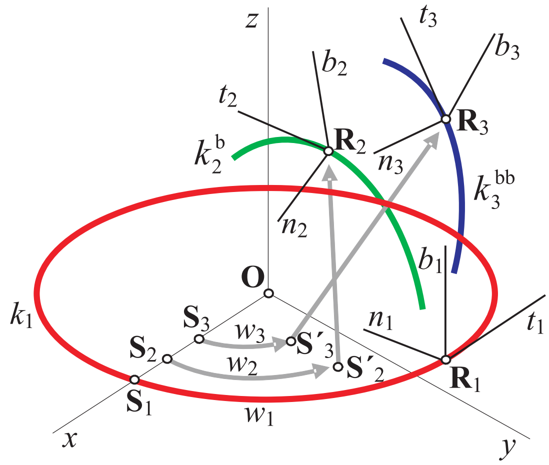

Fig. 2 Creation the curves k_1, k_2^g, k_3^{gh} and their trihedrons

The two-revolution cyclical surface of the type $P_2^g(u,v)$ is created by translation of the circle $c_2 = (R_2, r_2)$ along the curve k_2^g at an angular velocity $w_2 = m_1 v$, where the circle is always in the plane (n_2, b_2) if the index $g = t$, or in the plane (t_2, b_2) if $g = n$, or in the plane (t_2, n_2) if $g = b$ and its centre is the point $R_2 \in k_2^g$. We will create it so that the circle c_{02} determined by the vector function $c_{02}(u) = (0, r_2 \cos u, r_2 \sin u, 1)$ if $g = t$, or $c_{02}(u) = (r_2 \cos u, 0, r_2 \sin u, 1)$ if $g = n$, or $c_{02}(u) = (r_2 \cos u, r_2 \sin u, 0, 1)$ if $g = b$ we will transform into the circle c_2 in the coordinate system (R_2, t_2, n_2, b_2) using the matrix $M_2(w_2)$ by equations (5) (Fig. 4). The vector function of the cyclical surface of the type $P_2^g(u,v)$ is

$$P_2^g(u,v) = r_2(v) + c_{02}(u) \,.\, M_2(w_2),$$

$$u \in \langle 0, 2\pi \rangle. \; v \in \langle 0, 2\pi \rangle. \tag{9}$$

where $r_2(v)$ is the vector function of the curve k_2^g determined by (6).

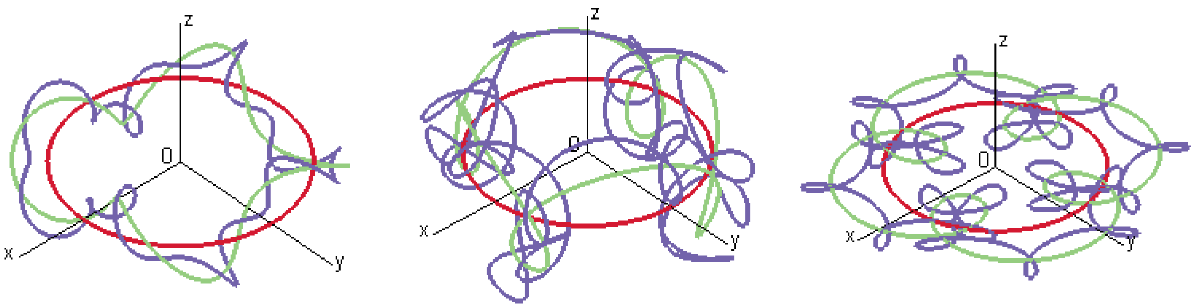

Fig. 3 Combinations of the curves k_1, k_2^t, k_3^{tt}, k_1, k_2^n, k_3^{nn} and k_1, k_2^b, k_3^{bb}.

3. Vector functions of cyclical surfaces of the type $P_1(u,v)$, $P_2^g(u,v)$, $P_3^{gh}(u,v)$

The cyclical surface of the type $P_1(u,v)$ is created by translation of the circle $c_1 = (R_1, r_1)$ in the plane (n_1, b_1) along the curve k_1 at an angular velocity $w_1 = v$. We will create it so that the circle c_{01} determined by the vector function $c_{01}(u) = (0, r_1 \cos u, r_1 \sin u, 1)$ will be transformed into the circle c_1 in the coordinate system (R_1, t_1, n_1, b_1) using the matrix $M_1(w_1)$ expressed by equations (5) (Fig. 4).

The vector function of the cyclical surface of the type $P_1(u,v)$ is

$$P_1(u,v) = r_1(v) + c_{01}(u) \,.\, M_1(w_1),$$

$$u \in \langle 0, 2\pi \rangle. \; v \in \langle 0, 2\pi \rangle. \tag{8}$$

where $r_1(v)$ is the vector function of the curve k_1 determined by equation (1). This surface is surface of torus.

The three-revolution cyclical surface of the type $P_3^{gh}(u,v)$ is created by translation of the circle $c_3 = (R_3, r_3)$ along the curve k_3 at an angular velocity $w_3 = m_2 m_1 v$, where the circle is always in the plane (n_3, b_3) if the index $h = t$, or in the plane (t_3, b_3) if $h = n$, or in the plane (t_3, n_3) if $h = b$ and its centre is the point $R_3 \in k_3^{gh}$. We will create it so that the circle c_{03} determined the by vector function $c_{03}(u) = (0, r_3 \cos u, r_3 \sin u, 1)$ if $h = t$, or $c_{03}(u) = (r_3 \cos u, 0, r_3 \sin u, 1)$ if $h = n$, or $c_{03}(u) = (r_3 \cos u, r_3 \sin u, 0, 1)$ if $h = b$ we will transform into the circle c_3 in the coordinate system (R_3, t_3, n_3, b_3) using the matrix $M_3(w_3)$ by equations (5). The vector function of the cyclical surface of the type $P_3^{gh}(u,v)$ is

$$P_3^{gh}(u,v) = r_3(v) + c_{03}(u) \,.\, M_3(w_3),$$

$$u \in \langle 0, 2\pi \rangle. \; v \in \langle 0, 2\pi \rangle. \tag{10}$$

where $r_3(v)$ is the vector function of the curve k_3^{gh} determined by (7).

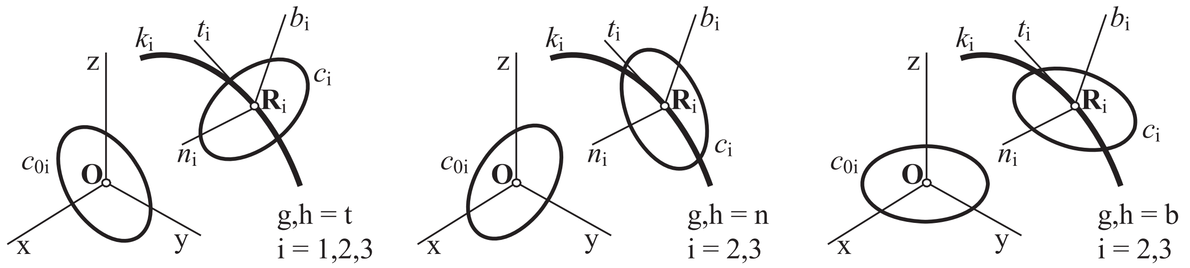

Fig. 4 Transformation of the circle c_{0i} into the circle c_i

4. Classification of cyclical surfaces of the type $P_2^g(u,v)$, $P_3^{gh}(u,v)$

The two-revolution cyclical surface of the type $P_2^g(u,v)$ can be classified according to the index g:

Table 1: Classification of cyclical surfaces of the type $P_2^g(u,v)$

g	*t*	*n*	*b*
	$P_2^t(u,v)$	$P_2^n(u,v)$	$P_2^b(u,v)$

The three-revolution cyclical surface of the type $P_3^{gh}(u,v)$ can be classified according to the index g and h:

Table 2: Classification of cyclical surfaces of the type $P_3^{gh}(u,v)$

g/h	*t*	*n*	*b*
t	$P_3^{tt}(u,v)$	$P_3^{tn}(u,v)$	$P_3^{tb}(u,v)$
n	$P_3^{nt}(u,v)$	$P_3^{nn}(u,v)$	$P_3^{nb}(u,v)$
b	$P_3^{bt}(u,v)$	$P_3^{bn}(u,v)$	$P_3^{bb}(u,v)$

5. Illustrations of cyclical surfaces of the type $P_1(u,v)$, $P_2^g(u,v)$, $P_3^{gh}(u,v)$

In Fig. 5 there are displayed three combinations of cyclical surfaces of the type $P_1(u,v)$, $P_2^t(u,v)$, $P_3^{tt}(u,v)$ in fig. a), $P_1(u,v)$, $P_2^t(u,v)$, $P_3^{tn}(u,v)$ in fig.b), $P_1(u,v)$, $P_2^t(u,v)$, $P_3^{tb}(u,v)$ in fig. c).

In Fig. 6 there are displayed three combinations of cyclical surfaces of the type $P_1(u,v)$, $P_2^n(u,v)$, $P_3^{nt}(u,v)$ in fig. a), $P_1(u,v)$, $P_2^n(u,v)$, $P_3^{nn}(u,v)$ in fig. b), $P_1(u,v)$, $P_2^n(u,v)$, $P_3^{nb}(u,v)$ in fig. c).

In Fig. 7 there are displayed three combinations of cyclical surfaces of the type $P_1(u,v)$, $P_2^b(u,v)$, $P_3^{bt}(u,v)$ in fig. a), $P_1(u,v)$, $P_2^b(u,v)$, $P_3^{bn}(u,v)$ in fig.b), $P_1(u,v)$, $P_2^b(u,v)$, $P_3^{bb}(u,v)$ in fig. c).

The surfaces mentioned above can be used in design practice as constructive or ornamental structural components.

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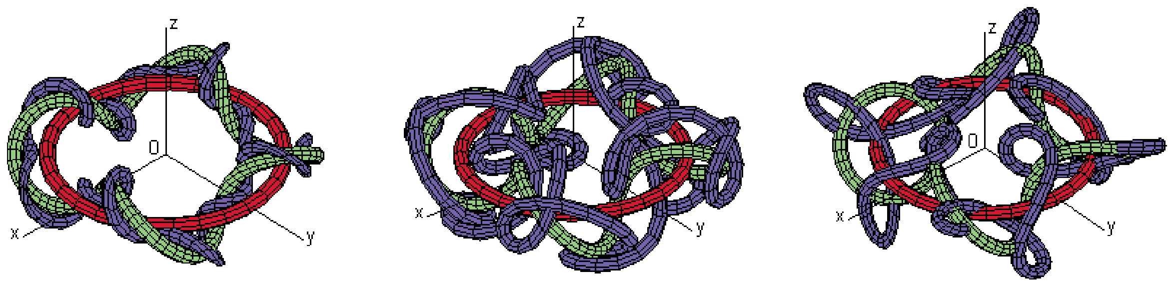
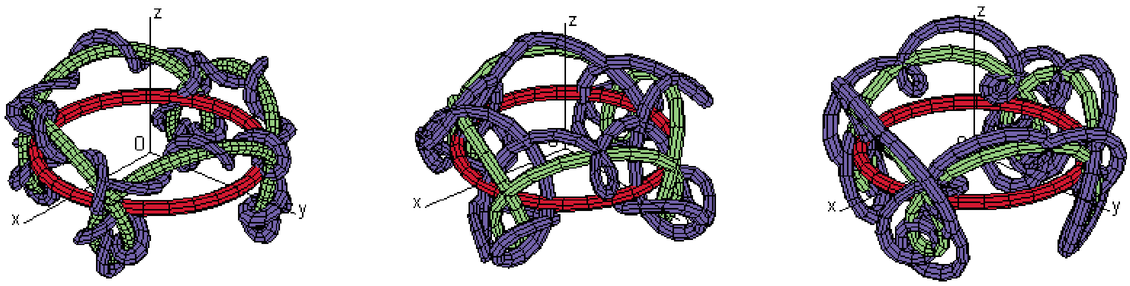


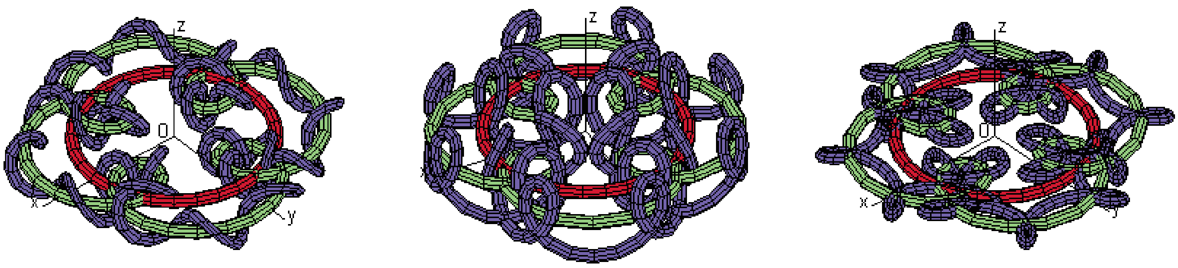
a) *b)* *c)*

Fig. 5 Combinations of cyclical surfaces for g = t



a) b) c)

Fig. 6 Combinations of cyclical surfaces for g = n



a) b) c)

Fig 7 Combinations of cyclical surfaces for g = b

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