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VIBRATION ANALYSIS BY THE WIGNER-VILLE TRANSFORMATION METHOD

This paper deals with description and application of the Wigner-Ville transformation for vibration analysis. This transformation belongs to the group of non-linear time-frequency processes. Thanks to its properties, it may be successfully used in the area of non-stationary and transitional signals describing various natural processes. The use in the field of the railway constructions testing represents a quite an interesting application area of the transformation. This paper contains mathematical analysis of the transformation, a case study and practical experience obtained and recommendations for its practical use.

Keywords: *non-linear time-frequency transform, cross-component, Heisenberg uncertainty principle, vibration, railway fastening*

1. Introduction

The information on any engineering, physical, or other phenomenon, is represented in the signal by changes over time of the current value of the quantity described by the signal. A large number of methods can be applied to the measured signal in the time domain.

In many applications, direct evaluation of the time-amplitude representation is neither easy nor advantageous. For this reason, the signal can be transformed from the time-domain into another one. In some cases, important information can be obtained from the frequency domain. Fourier transform-based methods are the most frequently used ones. Thus, Fourier transform, its modifications and some of the parametric methods are the well suited techniques for processing stationary (at best ergodic or periodic) signals. They can even be used to analyse the non-stationary signals if it is important to know only the frequency components contained in the entire signal. This, of course, gives no information on the time at which they occur. To localise such frequency components in time, some other transforming methods and other computational techniques have to be used. If the information sources from the time and frequency domains are combined one can use so called time-frequency transformations [1], [2]. This enables determination of the frequency as a function of time. The time-frequency transformations can be divided in two basic groups [3], [4]:

- Linear (including mainly short Time Fourier Transformation, Wavelet Transformation, etc.)
- Non-linear (including mainly Wigner-Ville transformation, quadratic Cohen transformations, affine and hyperbolic transformations, eventually some further special proceedings).

Advantages of the linear transformations are mainly the speed of calculation and satisfactory time-frequency distribution. The main disadvantage of the linear transformation is the fact that resulting differentiation in time and frequency is limited by the so-called Heisenberg principle of uncertainty. Hence, the component of the signal cannot be presented as a point in time-frequency space. It is therefore possible to state only its position inside the rectangle $\Delta t \cdot \Delta f$ in a given time-frequency area [5].

A characteristic feature of non-linear transformations is the fact that their resulting differentiation in time and frequency is not limited by the Heisenberg principle of uncertainty. This fact includes the high distinguishing ability in the time-frequency level that gives rise to "precise" localisation of important frequency components in time.

2. The Wigner-Villa's transformation

The Wigner´s distribution was proposed in 1932 by Professor Wigner in the field of quantum physics and about 15 years later, it was adapted for the area of signal analysis by the French scientist Ville. The Wigner-Ville transformation is defined for the time-frequency domain by relation [6], [7]

$$\mathrm{WVT_x}(\mathrm{t,f}) = \int_{-\infty}^{\infty} \mathrm{x}\left(t+\frac{\tau}{2}\right)\cdot \mathrm{x}^*\left(\mathrm{t}-\frac{\tau}{2}\right)\cdot \mathrm{e}^{(-\mathrm{j}\cdot 2\cdot\pi\cdot \mathrm{f}\cdot\tau)}\cdot \mathrm{d}\tau \qquad (1)$$

where '*' represents a complex conjunction, t is time, τ is shift along the time axis, x is time representation of the signal x(t) and $WVT_x(t,f)$ is a time-frequency representation of the input signal. Equation (1) shows that it is essentially the Fourier transformation of relation $x(t+\tau/2)\cdot x^*(t-\tau/2)$, so the functions $x(\tau/2)$ and the complex conjugate of $x^*(-\tau/2)$ at some point in time *t*. From Equation (1) it is also apparent that the Wigner-Ville transformation is a complex function in the time-frequency space. Similarly, one gets the equation for the calculation in the frequency domain. If the discrete data sequence is processed, it is necessary to modify the integral equation mentioned above (1) in the form of summation.

Unlike the other linear methods (for example the short time Fourier transformation), in which the resolution is limited by a window function, the Wigner-Ville spectrum provides a good

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resolution both in the frequency and in time domain. The important feature here is that the calculation is not limited by Heisenberg uncertainty principle. The Wigner-Ville distribution satisfies the time-frequency marginal according to Equation (2) [6], [8].

$$\int \mathrm{WVT}(t,\omega)d\omega = \left| x(t)^2 \right|$$

and $$\int \mathrm{WVT}(t,\omega)d\omega = \left| X(\omega)^2 \right| \tag{2}$$

where WVT(t,ω) is the Wigner-Ville transformation, x(t) is time signal, X(ω) is Fourier transformation of a time signal x(t), t is time and ω is angular frequency. Other properties of the Wigner-Ville distribution are - time shift invariant, frequency modulation invariant, synchronous invariant time shift and frequency modulation, time scaling.

Although the calculation of the coefficients of the Wigner-Ville transformation is not limited by the Heisenberg's uncertainty principle, certain problems may arise in calculating multicomponent signal, which is generated by the sum of two or more signals. The signal, which arises from additive combination (sum) of signals x_1 and x_2, is now considered according to equation

$$x(t) = x_1(t) + x_2(t) \tag{3}$$

For the Wigner Ville transformation subsequently applies the following relation

$$\begin{aligned}&\mathrm{WVT}(t,\omega) = W_{11}(t,\omega) + W_{22}(t,\omega) + W_{12}(t,\omega) + \\ &+ W_{21}(t,\omega) = W_{11}(t,\omega) + W_{22}(t,\omega) + \\ &+ 2\cdot \mathrm{Re}\{W_{12}(t,\omega)\}\end{aligned} \tag{4}$$

where the symbol "Re" stands for the real part and with keeping equality $W_{12} = W_{21}$ one obtains:

$$W_{12}(t,\omega) = \int_{-\infty}^{\infty} x_2\left(t+\frac{\tau}{2}\right)\cdot x_1^*\left(t-\frac{\tau}{2}\right)\cdot e^{(-j\cdot 2\cdot\pi\cdot f\cdot t)}\cdot d\tau \tag{5}$$

It is obvious that the Wigner-Ville transformation of the sum of the signals is not equal to the sum of the Wigner-Ville transformation of signals. There is an additional term 2·Re{W12(t,ω)}. This addition may be called interference or contribution to the cross-component. Based on the fact that the autocorrelation function is a bilinear operation on the processed signal and when it is formed, there are "false" contributions from the aforementioned cross-component in the final calculation of the time-frequency spectrum display, which then deteriorate the reproducibility of the view. For a better explanation of the problem defined above, consider now the signal composed of the two sine waves of frequencies f_1 = 5 Hz and f_2 = 20 Hz according to the equation

$$X(t) = A_1\cdot e^{j\cdot\omega_1\cdot t} + A_2\cdot e^{j\cdot\omega_2\cdot t} \tag{6}$$

The Wigner-Ville transformation of such a signal can be analytically expressed as

$$\begin{aligned}&\mathrm{WVT}(t,\omega) = 2\cdot\pi\cdot\sum_{i=1}^{2} A_i^2\cdot\delta\cdot(\omega-\omega_i) + \\ &+ 4\cdot A_1\cdot A_2\cdot\pi\cdot\delta\cdot(\omega-\omega_\mu)\cdot\cos(\omega_d\cdot t)\end{aligned} \tag{7}$$

where the symbol δ represents delta function, ω_μ and ω_d are the geometric center, or the distance between the two sinusoidal functions in the frequency plane, respectively, given by:

$$\omega_\mu = \frac{\omega_1+\omega_2}{2} \qquad \omega_d = \omega_2 - \omega_1 \tag{8}$$

Equation (7) fully corresponds to Equation (4) and it means that in the time-frequency plane the contributions of the examined signal are concentrated at the frequency ω_1, ω_2 and also to nonzero frequency ω_μ.

With the number of individual frequency components N, contained in the analyzed signal, one gets the total number of contributions from interference $N\cdot\frac{(N-1)}{2}$. To eliminate this effect certain adjustments of the Wigner-Ville transformation can be used for certain types of signals.

In principle, there are two reasons to modify the basic properties of the Wigner-Ville transformation. The first reason is that in practice it is not possible to integrate from -∞ to +∞, but the calculation can be carried out only for limited signals. The second reason is to try to eliminate the effect of the cross-components, which often oscillate heavily. Restrictions of both problems can be achieved by calculating the modified Wigner-Ville transformation (often called pseudo or smoothed) by equation [7], [9]

$$\begin{aligned}&\mathrm{PWVT}_X(t,f) = \int_{-\infty}^{\infty} h(\tau)\cdot x\left(t+\frac{\tau}{2}\right)\cdot \\ &\cdot x^*\left(t-\frac{\tau}{2}\right)\cdot e^{(-j\cdot 2\cdot\pi\cdot f\cdot\tau)}\cdot d\tau\end{aligned} \tag{9}$$

where the function h(τ) is a window function with maximum at τ = 0. Such a solution quite effectively suppresses interference (smoothed), but deteriorates the resolution. The smoothed Wigner-Ville transformation is therefore a certain compromise between smoothing and resolution. For the better explanation, consider a signal composed of the two sine waves according to the Equation (3). Suppose that, parameter α is set by expansion of function h(τ). The smoothed Wigner-Ville transformation of such a signal can be analytically expressed by the following relation:

$$\begin{aligned}&\mathrm{PWVT}_x(t,\omega) = \frac{1}{\sqrt{2\cdot\pi\cdot\alpha}}\cdot \\ &\cdot\left[A_1^2\cdot e^{-\frac{(\omega-\omega_1)}{2\cdot\alpha}} + A_2^2\cdot e^{-\frac{(\omega-\omega_2)}{2\cdot\alpha}}\right] + \\ &+ \frac{2\cdot A_1\cdot A_2}{\sqrt{2\cdot\pi\cdot\alpha}}\cdot\cos[(\omega_2-\omega_1)\cdot t]\cdot e^{-\frac{(\omega-(\omega-\omega_2))^2}{8\cdot\alpha}}\end{aligned} \tag{10}$$

For the practical presentation of these conclusions, a simulated signal, composed of the two sine waves of frequencies f_1 = 5 Hz and f_2 = 20 Hz and amplitude 0.5 V, based on equation (3), was used. It should be noted that the signal is technically stationary. Significant frequency components in the signal occur throughout the implementation.

At this signal the classical Wigner-Ville transformation was first applied, later the smoothed version ones. For the analysis, pictures (Figure 1 and Figure 2) were used, which consist of a trio of graphs. The top graph shows the time course of amplitude of changes in a physical quantity (in this case the voltage). The lower

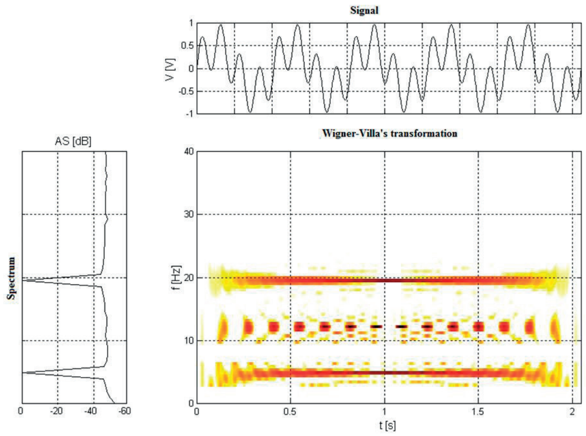

Figure 1 *The classical Wigner-Ville transformation of a simulated signal*

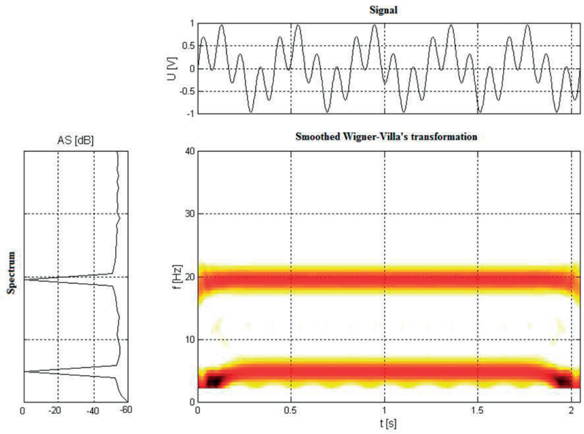

Figure 2 *The smoothed Wigner-Ville transformation of a simulated signal*

left graph shows the amplitude spectrum calculated by direct application of the Fourier transformation of the signal.

In the middle graph there is a 3D representation of the time-frequency course of amplitude spectrum computed by application of the Wigner-Ville transformation. The spectral data are shown in different colours. Note that the maximum value is marked as black. Figure 1 shows calculation results of classical the Wigner-Ville transformation signal according to Equation (7).

From the representation, it is evident that in the time-frequency spectrum in addition to the two basic frequencies of 5 Hz and 20 Hz, a cross component occurs also at the frequency of 12.5 Hz. Components are found in the spectrum, above that other interfering frequency, resulting from the finite length of the signal (sharp start and stop of the signal), or other discontinuities. From the middle graph of Figure 1, an extremely accurate localization of the two fundamental frequencies is particularly visible.

Figure 2 shows calculation results of the smoothed Wigner-Ville transformation from the signal according to Equation (10). As seen from the presented view, the influence of the cross component is suppressed by selecting an appropriate small parameter α in the function.

On the other hand, the middle graph in Figure 2 clearly shows that the localization of the two basic frequency components, in the frequency domain, is obviously worse. This is due to the higher blur on the frequency axis.

3. Case study

The measurements were made on a test sample of a rail fastening. This sample was composed of UIC60 structural rail with elastic fastening Vossloh Skl14, mounted on a B91P concrete sleeper. For the experimental investigation of the dynamic properties of the test sample, a method based on the measurement of a mechanical shock response was used. The mechanical shock was excited by a special hammer, which had a force transducer, in the radial direction to the rail head. The response was measured by acceleration sensors on the foot of the rail and the sleeper seat, as shown in Figure 3.

Figure 3 *View of the rail fastening sample*

Frequency response functions were calculated from the response signals due to normalization to the excitation signal. Those were recalculated by the inverse Fourier Transform into the time domain. In this way, the normalized time course of oscillation acceleration was obtained. This method, including instrumentation, is described in detail in the literature [8], [10]. Analysis in time, frequency and time-frequency domain was used to evaluate the measured data. The Fourier transform was used for the analysis in the frequency domain. A procedure, based on application of the Wigner-Ville transformation, was used for analysis in the time-frequency domain.

Time histories of the impulse response function, recorded by accelerometers, located on the rail foot, are depicted in the upper graph of Figure 4. The left graph of Figure 4 shows the amplitude spectrum calculated by applying the Fourier transform. There are six distinct frequencies (200 Hz, 600 Hz, 1.7 kHz, 1.9 kHz, 3.2 kHz and 3.3 kHz).

The time-frequency amplitude spectrum, estimated by application of the Wigner-Ville transformation from the impulse response function, is depicted in the middle graph in Figure 4. As shown in this graph, the time history of important frequency components essentially differs. As can be seen from this graph, the time occurrence of significant frequency components varies considerably. The 1.9 kHz frequency component acquires the highest values for a relatively long time (relative to other

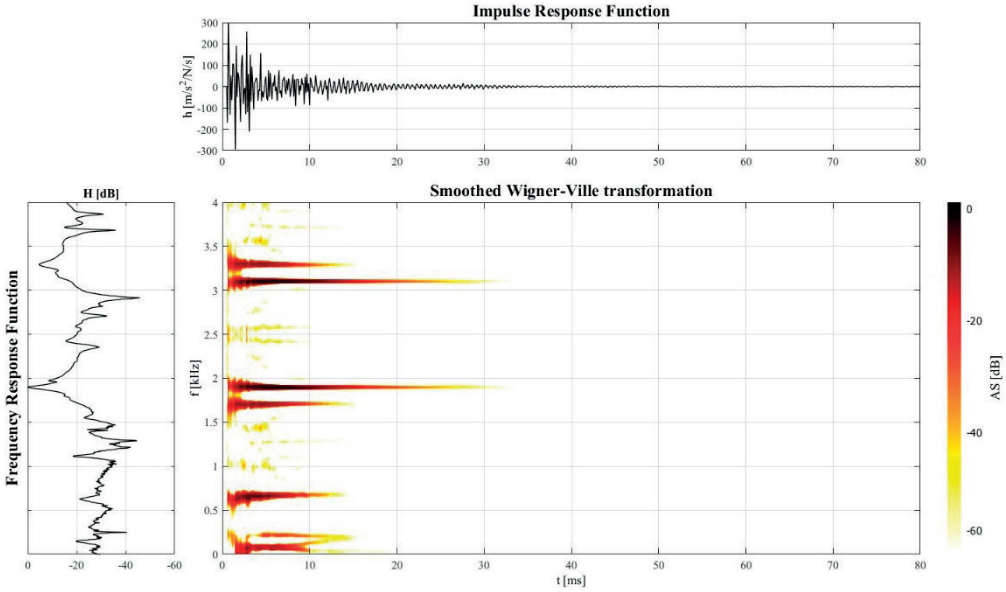

Figure 4 *Sensor located on the rail foot, time-frequency analysis by the Wigner-Ville transformation method*

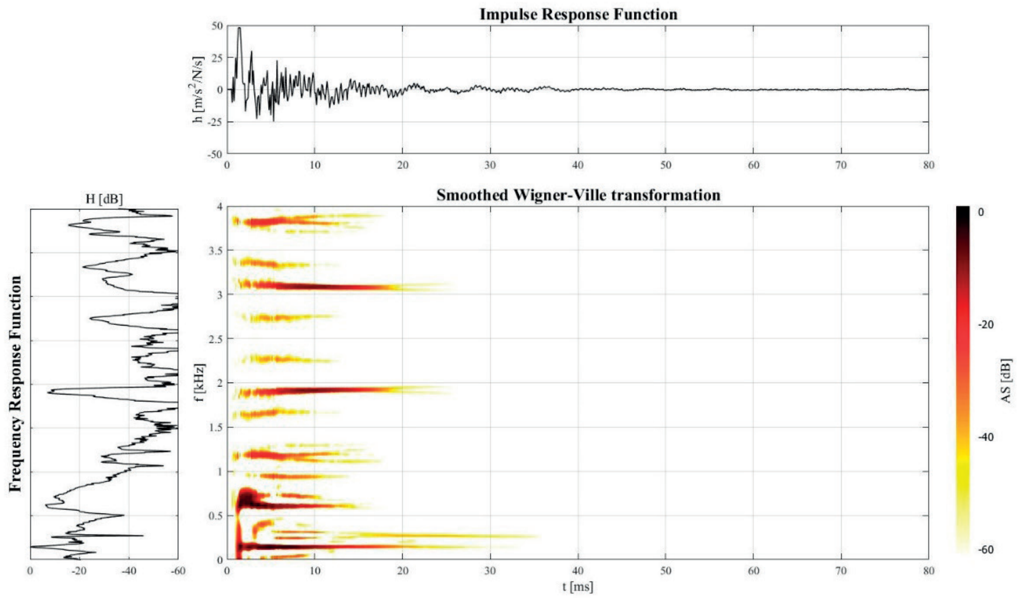

Figure 5 *Sensor located on concrete sleeper, time-frequency analysis by the Wigner-Ville transformation method*

frequency components). It occurs in the signal almost in its entirety, i.e. about 32 ms (at attenuation of up to 40 dB). The second most important component is the frequency 3.2 kHz with a duration of 30 ms. Others no,le frequencies of 1.7 kHz and 3.3 kHz are in the signal for the time of 2 ms up to 15 ms.

The signal taken by the second sensor, located on the concrete sleeper seat, has a different character. From the time recording (see the upper graph of Figure 5), it is clear that the maximum acceleration amplitude is lower than the signal from the sensor on the foot of the rail (Figure 4), due to the waving process via the fastening rail, rail pad and sleeper to the accelerometer sensor.

The course of the amplitude spectrum (left graph of Figure 5) differs significantly from the characteristics measured by the sensor located on the foot of the rail. The most significant components occur at the lower frequencies than those captured by the sensor at the foot of the rail, i.e. in the range of 200 Hz to 2 kHz, and there is a higher number of them in this interval. The same conclusion is provided in the middle graph of Figure 5, which presents the time-frequency representation of the Wigner-Ville coefficients. The longest component is 200 Hz with a duration of up to 34 ms. The highest values are on frequencies 600 Hz and 700 Hz with a duration of about 15 ms. A very interesting waveform has a frequency of 1.9 kHz, which occurs in the time interval of 2 ms to 24 ms. It is the same with the frequency of 400 Hz, which occurs between 5 ms and 35 ms.

4. Conclusion

The Wigner-Ville transformation offers a comprehensive tool for analysing the non-stationary signals, in particular. Characteristic feature of presented transformation is the fact that its resulting distinguishing in time and frequency is not limited by the Heisenberg uncertainty principle. This fact includes the high distinguishing ability in the time-frequency plane that gives rise to a "precise" localisation of important frequency components in time. This method gives a fast and accurate localisation of frequency components included in the measured signal.

The existence of the "false" interference frequency components may be a certain disadvantage of the Wigner-Ville transformation. However, their influence can be effectively defused by using the so-called smoothed version, when the properties of a given transformation are affected by the use of suitable local window function. The result is a compromise solution where one obtains the reasonable time and frequency resolution when interferences are suppressed.

From this point of view, the Wigner-Ville transformation is highly applicable in the area of railway construction upon noise, vibration and strain analyses. It is possible to apply this method successfully not only on samples of several constructions of railway and tramway superstructure, but directly in the field on real tracks, as well.

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