NUMERICAL AND ANALYTICAL INVESTIGATION OF STABILITY OF THE REINFORCED PLATE

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Resume

The work is aimed at the construction of an algorithm for studying the equilibrium states of a reinforced plate near critical points, using the first (cubic terms) nonlinear terms of the potential energy expansion. Using geometrically nonlinear analysis of displacement, deformation and stress fields, the Eigenforms of buckling were calculated and bifurcation solutions and solutions for equilibrium curves with limit points were constructed depending on the initial imperfections.

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1 Introduction

Interest in the work of compressed reinforced plates appeared long ago, however, only starting with the work of Koiter and Kuiken [1], Koiter and Pignataro [2], van der Neut [3], Tvergaard [4], Hunt [5] and at a later time Manevich [6-8], systems of equilibrium equations were obtained taking into account geometric nonlinearity, which make it possible to analyze the bearing capacity of the mentioned plate either taking into account the total deflection, or taking into account the interaction of this deflection with local waveforms in the ribs or in the plate. Geometrically nonlinear equations for describing the loss of stability of the considered plates, taking into account the interaction of forms, were first presented by Koiter and Pignataro [2], Tvergaard focused his attention on study of plates with double critical loads for the total deflection and for wave formation in the plate [4].

In his work, important and general results were obtained for the calculation of reinforced plates of sufficiently large width and regular arrangement of ribs. For this, he isolated a regular T-shaped fragment, which was investigated in more detail; some of the results obtained by Tvergaard were used by Hunt [5] to construct the bifurcation surface of the homeoclinic bifurcation point corresponding to the catastrophe of the hyperbolic umbilic.

Tvergaard's studies were continued by authors of this work [9-11], who, using the FEM, numerically, studied the effect of initial imperfections on the bearing capacity of a reinforced plate in the case of multiple and multiple critical loads. In these works, it was found that, indeed, with simultaneous buckling in the total deflection and wave formation in platinum, in the case of coincidence of the critical loads, a double bifurcation is realized as a homeoclinic point of a hyperbolic umbilic. If buckling simultaneously occurs along the general deflection and wave formation in the ribs, then the double semi-symmetric point is an anticlinic bifurcation corresponding to the catastrophe of the elliptical umbilic [11]. Important results, using equations of the first and second geometrically nonlinear approximations (taking into account cubic and quartic terms in the expansion of potential energy), were obtained in the 80s by Manevich [7-8]. He was able to establish that, limiting himself to the first approximation (only cubic terms are taken into account), it is possible to obtain acceptable estimates of the bearing capacity of a compressed reinforced plate if the critical loads of the wave formation in its elements are close to or exceed the critical buckling load according to the general deflection scheme. If the waveform loads are significantly less than the critical load of the total deflection, then taking into account additional terms of the fourth order in decomposition of the potential energy makes it possible to increase the maximum load loss of the bearing capacity of the reinforced plate (on average) by 30-40%. However, taking into account quartic terms significantly increases the computational complexity due to the need to determine the corrections of the second approximation to eigenforms. In this article, the authors limited themselves to taking into account only the cubic terms in the expansion of the fields of displacements, deformations and stresses, since in design practice such ratios of the geometric parameters of a reinforced plate are used, at which the critical loads of the wave formation in a plate or in the ribs are significantly higher than the critical load of the total deflection.

2 Semi-analytical solution

Here is considered consider a plate hinged at its ends, supported on one side by a regular set of thin rectangular ribs. The longitudinal edges are free. The plate is compressed by centrally applied forces. The plate material is infinitely elastic. This allows to investigate the stability of the reinforced plate by analyzing the equilibrium of one regular T-shaped fragment (by analogy with Tvergaard and Manevich) equidistant with the rest of the T-shaped fragments. Strains and Hooke's law are expressed as follows:

$$\varepsilon = L_1(U) + \frac{1}{2}L_2(U), \tag{1}$$

$$\sigma = H(\varepsilon), \tag{2}$$

where: L_1, H - is a linear operator,
L_2 -is a quadratic operator,
$L_2\ (U + V) = L_2\ (U) + 2\ L_{11}\ (U,\ V) + L_2\ (V)$, in this expression L_{11} - bilinear operator.

The total potential energy of the reinforced plate with the retention of terms no higher than the fourth order has the form:

$$\begin{aligned}\Pi_э = a_0 + \frac{1}{2}\sum_s a_s\Big(1-\frac{\lambda}{\lambda_s}\Big)\xi_s^2 + \frac{1}{3}\sum_i\sum_j\sum_k \times \\ \times a_{ijk}\xi_i\xi_j\xi_k + \frac{1}{4}\sum_i\sum_j\sum_k\sum_l a_{ijkl}\xi_i\xi_j\xi_k\xi_l - \\ \sum_s a_s\xi_s\overline{\xi}_s\frac{\lambda}{\lambda_s},\end{aligned} \tag{3}$$

where: ξ - are normalized Eigenforms (the amplitude of the deviation of the Eigenmode relative to the plate thickness,
$\overline{\xi}_s$ - imperfections in the s-th form of buckling (s = 1, 2, … ..n).

Equilibrium equations are:

$$\Big(\frac{\partial \Pi_э}{\partial \xi_1}\Big) = 0; \Big(\frac{\partial \Pi_э}{\partial \xi_2}\Big) = 0; \ldots \Big(\frac{\partial \Pi_э}{\partial \xi_s}\Big) = 0\,, \tag{4}$$

$$\begin{aligned}a_s\Big(1-\frac{\lambda}{\lambda_s}\Big)\xi_s + \sum_i\sum_j\sum_k a_{ijk}\xi_i\xi_j + \\ + \sum_i\sum_j\sum_k\sum_l a_{ijkl}\xi_i\xi_j\xi_k = \sum_s a_s\xi_s\overline{\xi}_s\frac{\lambda}{\lambda_s}.\end{aligned} \tag{5}$$

If s = 2 (the total deflection (i = 1)) and the local form of wave formation (i = 2) are taken into account, then the potential energy, taking into account the interaction of these forms, will be written as follows:

$$\begin{aligned}\Pi_э = a_0 + \frac{1}{2}a_1\Big(1-\frac{\lambda}{\lambda_1}\Big)\xi_1^2 + \frac{1}{2}a_2\Big(1-\frac{\lambda}{\lambda_2}\Big)\xi_2^2 + \\ + \frac{1}{3}a_{111}\xi_1^3 + a_{122}\xi_1\xi_2^2 + \frac{1}{4}a_{1111}\xi_1^4 + \frac{1}{4}a_{2222}\xi_2^4 + \\ + \frac{1}{2}a_{1122}\xi_1^2\xi_2^2 - \frac{\lambda}{\lambda_1}a_1\overline{\xi}_1\xi_1\frac{\lambda}{\lambda_2}a_2\overline{\xi}_2\xi_2.\end{aligned} \tag{6}$$

Accordingly, one obtains a simplified version of the equilibrium equations:

$$\begin{aligned}a_1\Big(1-\frac{\lambda}{\lambda_1}\Big)\xi_1 + a_{111}\xi_1^2 + a_{122}\xi_2^2 + a_{1111}\xi_1^3 + \\ + a_{1122}\xi_1\xi_2^2 = \frac{\lambda}{\lambda_1}a_1\overline{\xi}_1,\end{aligned} \tag{7}$$

$$\begin{aligned}a_2\Big(1-\frac{\lambda}{\lambda_2}\Big)\xi_2 + 2a_{122}\xi_1\xi_2 + a_{122}\xi_1^2\xi_2 + \\ + a_{2222}\xi_2^3 = \frac{\lambda}{\lambda_2}a_2\overline{\xi}_2.\end{aligned} \tag{8}$$

For the first nonlinear approximation equations, only the cubic terms are retained.

$$a_1\Big(1-\frac{\lambda}{\lambda_1}\Big)\xi_1 + a_{111}\xi_1^2 + a_{122}\xi_2^2 = \frac{\lambda}{\lambda_1}a_1\overline{\xi}_1\,, \tag{9}$$

$$a_2\Big(1-\frac{\lambda}{\lambda_2}\Big)\xi_2 + 2a_{122}\xi_1\xi_2 = \frac{\lambda}{\lambda_2}a_2\overline{\xi}_2\,. \tag{10}$$

If one divides each equation by coefficients a_1 and a_2, then one gets equations in the form given by Tvergaard [4] $(\lambda_1 = \lambda_2 = \lambda_c)(\lambda_1 = \lambda_2 = \lambda_c)$:

$$\Big(1-\frac{\lambda}{\lambda_1}\Big)\xi_1 + d_1\xi_1^2 + d_2\xi_2^2 = \frac{\lambda}{\lambda_1}\overline{\xi}_1\,, \tag{11}$$

$$\Big(1-\frac{\lambda}{\lambda_1}\Big)\xi_2 + d_3\xi_1\xi_2 = \frac{\lambda}{\lambda_2}\overline{\xi}_2\,, \tag{12}$$

where: $d_1 = \dfrac{a_{111}}{a_1}, d_2 = \dfrac{a_{122}}{a_1}, d_3 = \dfrac{2a_{122}}{a_2}$.

To determine the coefficients a_{ijk} and d_1, d_2 and d_3 it is necessary to calculate some definite integrals over the rectangular areas of the mentioned T-element of the reinforced plate. Expressions for these integrals: $a_1 = -\lambda_1\{\sigma^0 L_2(U^{(1)})\}$, $a_2 = -\lambda_2\{\sigma^0 L_2(U^{(2)})\}$, $a_{111} = \frac{3}{2}\{\sigma^1 L_2(U^{(1)})\}$ and $a_{112} = \{\sigma^1 L_2(U^{(2)})\} + \{\sigma^2 L_{11}(U^{(1)}, U^{(2)})\}$ are given in the work of Tvergaard [4] and in a slightly modified form in the work of Manevich [8]. Equations (9) and (10) include the values of the critical parameters λ_1 and λ_2, which correspond to the general shape of bending and local wave formation in the plate or ribs. These quantities are found by solving a linear eigenvalue problem. Tvergaard solved a homogeneous boundary value problem for a system of biharmonic equations and eight boundary conditions for conjugation along the line of contact of the rib with the plate:

$$D\,\Delta\Delta w_c = \lambda_c N_X^0 w_{,xx}, \tag{13}$$

$$\frac{1}{Eh}\Delta\Delta F_C = 0\,. \tag{14}$$

These conditions express the relationship between the forces and displacements at the edges of the left-hand and right-hand parts of the plate in relation to the rib [8]. In this case, it is assumed that the total deflection develops along a curve close to a sinusoid corresponding to the buckling of the Eulerian rod. Eight unknown arbitrary constants are found from a system of homogeneous equations, the matrix of coefficients of which is given in the appendix to the work of Tvergaard [4]. Manevich solved the same problem variationally, using the Rayleigh ratio. By setting a different number of half-waves, he found the smallest value of the critical loads of wave formation in the plate or in the ribs.

In Equations (11) and (12), on the right-hand sides, there are values of the initial imperfections $\overline{\xi_1}$ and $\overline{\xi_2}$. They correspond to buckling shapes in terms of the total deflection and in terms of the shape of local wave formation. If both of these imperfections are nonzero, then the critical point is the limiting point. For a homogeneous bifurcation problem $(\overline{\xi_1} = \overline{\xi_2} = 0)$, the simplest solutions correspond to the case of multiple loads $(\lambda_1 = \lambda_2 = \lambda_c)$. Bifurcation solutions for simple loads $(\lambda_1 \neq \lambda_2)$ turn out to be somewhat more complicated. If the right-hand side of Equation (12) is zero and the right-hand side of Equation (11) has an imperfection $(\overline{\xi_1} \neq 0)$ proportional to the load parameter, then the bifurcation problem is implemented as a search for the critical value of the load at which a plate with a developing initial general deflection loses stability in the form of wave formation in the plate or in the ribs.

Let the solutions of the simplest problem for a double semi-symmetric bifurcation point be considered, corresponding to solutions of homogeneous equations of Tvergaard's type. For the general deflection and for the solution describing wave formation in the plate, the mentioned solutions can be written as follows:

$$\xi_1 = -\frac{1}{d_3}\left(1 - \frac{\lambda}{\lambda_c}\right), \tag{15}$$

$$\xi_2 = \mp\sqrt{\left(\frac{d_3 - d_1}{d_2 d_3^2}\right)}\left(1 - \frac{\lambda}{\lambda_c}\right) = \mp\xi_1\sqrt{\frac{d_3 - d_1}{d_2}}\,. \tag{16}$$

From the last relations it can be seen that the total deflection of the relative amplitude ξ_1 is an unconnected deformation. It depends on one coordinate only, while the wave formation with an amplitude ξ_2 is linearly related to the amplitude of the total deflection. If this amplitude ξ_1 is equal to zero, then the amplitude ξ_2 is also equal to zero. For the values of the coefficients of the equations calculated by Tvergaard d_1 = -0.0193, d_2 = 0.6731 and d_3 = 0.1138. The ratio between ξ_2 and ξ_1 is $\sqrt{\frac{d_3 - d_1}{d_2}} = 0.374$. Note that all the three coefficients d_1, d_2, and d_3 are negative, which is possible only with a homeoclinic bifurcation point (a variant of the catastrophe of a hyperbolic umbilic). This situation arises during wave formation in a plate. In the case of simple eigenvalues $(\lambda_1 \neq \lambda_2)$ the bifurcation solutions of the system of nonlinear equations with zero right-hand sides $\xi_1 = \xi_2 = 0$ turn out to be more complicated:

$$\xi_1 = -\frac{1}{d_3}\left(1 - \frac{\lambda}{\lambda_2}\right), \tag{17}$$

$$\xi_2 = \mp\frac{1}{d_3}\sqrt{\left(1 - \frac{\lambda}{\lambda_2}\right)\left(\frac{d_3}{d_2}\left(1 - \frac{\lambda}{\lambda_1}\right) - \frac{d_1}{d_2}\left(1 - \frac{\lambda}{\lambda_2}\right)\right)} = \mp\xi_1\sqrt{\frac{d_3\left(1 - \frac{\lambda}{\lambda_1}\right)}{d_2\left(1 - \frac{\lambda}{\lambda_2}\right)} - \frac{d_1}{d_2}}\,. \tag{18}$$

Singular points (bifurcation points or limit points) appear on the equilibrium curves of the reinforced plate when the Hessian matrix degenerates in critical equilibrium $(\det H(\xi_1, \xi_2, \lambda) = 0)$. For Equations (11), (12) this matrix has the form:

$$H = \begin{bmatrix} \left(1 - \frac{\lambda}{\lambda_1}\right) + 2d_1\xi_1 & 2d_2\xi_2 \\ d_3\xi_2 & \left(1 - \frac{\lambda}{\lambda_2}\right) + d_3\xi_1 \end{bmatrix}. \tag{19}$$

If a problem is considered in which the initial imperfection in the total deflection is specified $(\overline{\xi_1} \neq 0)$ and there is no initial wave formation $(\overline{\xi_2} = 0)$, then in this case the solution depends only on the coordinate $\xi_1(\xi_2 = 0)$ and the condition of equality to zero of the determinant $(\det H(\xi_1, \xi_2, \lambda) = 0)$ is reduced to the fulfillment of one of two relations:

$$h_{11} = \left(1 - \frac{\lambda}{\lambda_1}\right) + 2d_1\xi_1 = 0\,,\ \text{or} \tag{20}$$

$$h_{22}\left(1 - \frac{\lambda}{\lambda_2}\right) + d_3\xi_1 = 0\,. \tag{21}$$

The most important condition is Equation (21). $(h_{22}(\lambda, d_3, \xi_1) = 0)$. In this case, the equilibrium Equation (12) $\left(\left(1 - \frac{\lambda}{\lambda_2}\right) + d_3\xi_1\right)\xi_2 = 0$ is fulfilled in two versions:

a) $\xi_2 = 0$, but $\left(\left(1 - \frac{\lambda}{\lambda_2}\right) + d_3\xi_1\right) \neq 0$.

Hence $(h_{22}(\lambda, d_3, \xi_1) \neq 0), \det H(\xi_1, \xi_2, \lambda) \neq 0)$, which corresponds to regular points on the equilibrium curve $\mu_1(\xi_1, \overline{\xi_1}), \left(\mu_1 = \frac{\lambda}{\lambda_1}\right)$.

b) $\xi_2 = 0$ and $\left(\left(1 - \frac{\lambda}{\lambda_2}\right) + d_3\xi_1\right) = 0$.

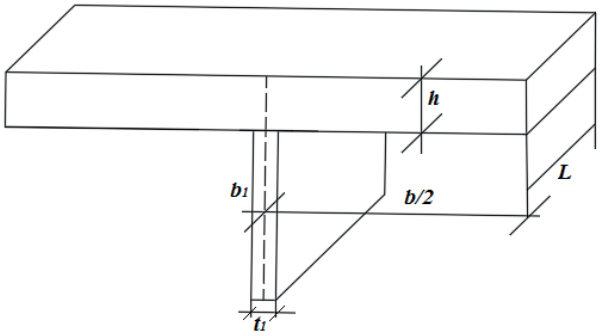

Figure 1 *Cross-section of a T-shaped fragment; here b - is the distance between the ribs in the axes, h - is the plate thickness, b_1 - is the rib height, t_1 - is the rib thickness, L - is the length of the reinforced plate, boundary conditions: hinged support along the short sides, along the longitudinal edges, fastening of the movable termination type*

Hence $h_{22}(\lambda, d_3, \xi_1) = 0, \det H(\xi_1, \xi_2, \lambda) = 0)$ and the corresponding point of the curve $\mu_1(\xi_1, \overline{\xi_1})$ is a point of singular equilibrium. Whether this point is a limit point or a bifurcation point depends on whether the value $\mu_1(\xi_1^{cr})$ is a local extremum or not. If the value $\mu_1(\xi_1^{cr})$ is not a local extremum, then the singular point is a bifurcation point (in this problem, symmetric and unstable).

c) $\xi_2 \neq 0, \left(\left(1 - \frac{\lambda}{\lambda_2}\right) + d_3\xi_1\right) = 0$ moreover $h_{22}(\lambda, d_3, \xi_1) = 0$.

However, $\det H(\xi_1, \xi_2, \lambda) = -2d_2 d_3 \xi_2^2 \neq 0$ and the corresponding equilibrium is regular. The greatest compression load is achieved here, since the post-bifurcation equilibrium turns out to be unstable. The values of the bifurcation load $\mu_1^{cr}\left(\mu_1 = \frac{\lambda}{\lambda_1}\right)$ are determined from the quadratic equation, compiled taking into account the relations:

$$h_{22}(\lambda, d_3, \xi_1) = 0, \xi_1 = -\frac{1 - k\mu_1}{d_3} = \frac{k\mu_1 - 1}{d_3}, \quad k = \frac{\lambda_2}{\lambda_1}, \tag{22}$$

$$\mu_1^2(k^2 d_1 - d_3 k) + \mu_1(d_3 + d_3 k - 2kd_1 - d_3^2 \overline{\xi_1}) + d_1 - d_3 = 0. \tag{23}$$

In the problem solved by Tvergaard [4], the critical load is twofold (at the same time there is a general buckling and wave formation in the plate). By specifying a nonzero initial imperfection in the total deflection $\overline{\xi_1} \neq 0)$, it will be possible to plot the sensitivity curve of critical loads at the limiting points depending on the value of the initial deflection $\overline{\xi_1}$. Calculations according to Equation (23) for $k - 1$ give:

$$\begin{aligned} \overline{\xi_1} &= 0.2 \quad \mu_1^*(0,2) = 0.8476, \\ \overline{\xi_1} &= 0.5 \quad \mu_1^*(0,5) = 0.7703, \\ \overline{\xi_1} &= 1.0 \quad \mu_1^*(1,0) = 0.692. \end{aligned} \tag{24}$$

These results coincide with the values of ultimate loads with imperfections of the total deflection on the curves constructed by Tvergaard [4]. Consider the bifurcation problem in the presence of an initial imperfection in the form of a general deflection $(\overline{\xi_1} \neq 0)$. Its solution gives a critical waveform load in the relatively weak ribs of the reinforced plate (Figure 1). In the graphical form, results for this problem, with various initial imperfections, are presented in the work of Manevich ([8], ch. III, Figure 3.3). The investigated plate had the following dimensionless parameters:

$$\frac{b}{h} = 25, \frac{b_1}{t_1} = 20, \frac{t_1 b_1}{bh} = \frac{1}{5}, \frac{L}{b} = 5, k = \frac{\lambda_1}{\lambda_2} = 1.4852.$$

For the plate in Figure 1, the values of the coefficients of the Equations, (11) and (12): $d_1 = -0.009, d_2 = 1.024$ and $d_3 = 0.975$, taken from [8].

Note that this rib plate contains relatively weak elongated cantilever plates. Loss of stability by wave formation should be expected precisely in compressed ribs, and not in the plate. This is indicated by the signs of the coefficients $d_1 < 0$ (this is always), $d_2 > 0$ and $d_3 > 0$. If all the coefficients $d_i < 0$, then the wave formation should be expected in the plate. It is assumed that the relative amplitude of the initial total deflection is $\overline{\xi_1} = -0.5$. Substituting all the given data into Equation (23) one gets:

$$\mu^2 - 1.9927\mu + 0.67033 = 0. \tag{25}$$

The smallest root $\mu_1^{cr} = 0.428$ determines the load of the bifurcation of wave formation in the compressed ribs of the plate with an additional total deflection:

$$\xi_1 = -\frac{1 - k\mu_1}{d_3} = -0.3728. \tag{26}$$

This bifurcation is symmetrical and unstable, since in the presence of the initial additional wave formation in the edges $(\overline{\xi_2} \neq 0)$ loss of stability occurs at the limiting points. Note that in this case the equilibrium curve is spatial, since the load μ_1 depends on two coordinates ξ_1 and ξ_2.

The calculated value of the wave-forming load $\mu_1^{bif} = 0.428$ coincides with the critical point in table 3.5 in [8]. Having performed the similar calculations for other initial deflections, a curve is obtained of

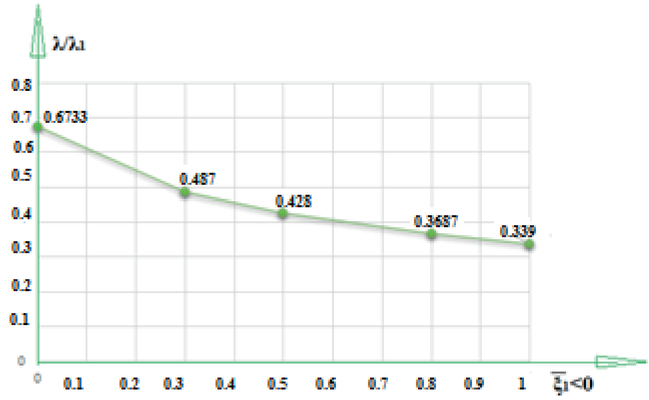

Figure 2 *Bifurcation curve*

dependence of the critical load for the wave formation bifurcation in the ribs (Figure 2) on value of the initial imperfection in a form of a general deflection.

From the graph in Figure 2 can be seen that the value of the critical load of wave formation in the ribs strongly decreases with an increase in the initial imperfection $\overline{\xi_1}$; when $\overline{\xi_1} = 1$, then the wave formation load is approximately twice smaller than the eigenvalue from the solution of the linear homogeneous problem $\left(\text{for } \overline{\xi_1} = 0, \lambda_2 = 0.6733\lambda_1, k = \frac{1}{0.6733} = 1.4852\right)$.

If in Equations (11) and (12) the right-hand sides are not equal to zero, then the problem arises of determining the coordinates of the limit point (ξ_1^*, ξ_2^*) and the value of the maximum load μ_1^*.

$$(1-\mu_1)\xi_1 + d_1\xi_1^2 + d_2\xi_2^2 = \mu_1\overline{\xi_1}, \tag{27}$$

$$(1-\mu_2)\xi_2 + d_3\xi_1\xi_2 = \mu_2\overline{\xi_2}. \tag{28}$$

From Equation (27) $\xi \neq 0$ is expressed as:

$$\xi_2 = \frac{k\mu_1\overline{\xi_2}}{1 - k\mu_1 + d_3\xi_1}. \tag{29}$$

Substituting the resulting expression into Equation (28) gives:

$$(1-\mu_1)\xi_1 + d_1\xi_1^2 + \\ + \frac{(k\mu_1\overline{\xi_2})^2 d_2}{((1-k\mu_1)+d_3\xi_1)^2} = \mu_1\overline{\xi_1}. \tag{30}$$

As a result, freed from the denominator, to determine the coordinates of the equilibrium point ξ_1, (at a fixed value of the load μ_1) one obtains an equation of the fourth order with respect to ξ_1. After some transformations, it can be written as:

$$d_1d_3\xi_1^4 + [2d_1d_3(1-k\mu_1) + d_3^2(1-\mu_1)]\xi_1^3 + \\ + [2d_3(1-\mu_1)(1-k\mu_1) + d_1(1-k\mu_1)^2 - \\ - d_3^2\mu_1\overline{\xi_1}]\xi_1^2 + [(1-\mu_1)(1-k\mu_1)^2 - \\ - 2d_3(1-k\mu_1)\mu_1\overline{\xi_1}]\xi_1 + (k\mu_1)^2 d_2\xi\overline{\xi}_2^2 - \\ - (1-k\mu_1)^2\mu_1\overline{\xi_1} = 0. \tag{31}$$

The condition that the determinant of the Hessian matrix is equal to zero Is added to this equation:

$$\det H(\mu_1,\xi_1,\xi_2) = [(1-\mu_1) + 2d_1\xi_1]\times \\ \times[(1-k\mu_1) + d_3\xi_1] - 2d_2d_3\xi_2^2 = 0. \tag{32}$$

After excluding the coordinates ξ_2 from the corresponding expression, we will have:

$$\det H(\mu_1,\xi_1,\lambda) = (1-\mu_1) + 2d_1\xi_1(1-k\mu_1) + \\ + d_3\xi_1 - \frac{d_1d_3(k\mu_1)^2\overline{\xi}_2^2}{(1-k\mu_1+d_3\xi_1)^2} = 0. \tag{33}$$

As a result, one gets the two equations for variables μ_1 and ξ_1. However, their joint solution is quite difficult. Tvergaard was looking for limit points by directly constructing the equilibrium curve, while Manevich used algorithms for finding the extrema of a function of two variables.

The method of the step-by-step increase in the load, is used in this work, followed by solution of the equilibrium equation to approximately obtain the coordinates of the limiting point and the corresponding maximum load. As an example, the coordinates of the limiting point and the corresponding load are determined for the above stability problem of a reinforced plate with wave formation in the ribs. Substitution of numerical values into Equations (31), after some simplifications, leads to the following equation of the fourth order:

$$f(\xi_1) = -0.008773\xi_1^4 + 0.67376\xi_1^3 + 0.94993\xi_1^2 + \\ + 0.405\xi_1 + 0.0495 = 0. \tag{34}$$

With a fixed value of the force parameter $\mu_1 = 0.28$, one finds the minimal modulo negative root of this equation. For that is used a method of the non-singular extensions, proposed by Manuilov in 1971 [12].

$$\xi_1^{m+1} = \xi_1^m \pm \frac{|f(\xi_1^{(m)})|}{\sqrt{[f'^2(\xi_1^{(m)}) - f(\xi_1^{(m)})f''(\xi_1^{(m)})]}}. \tag{35}$$

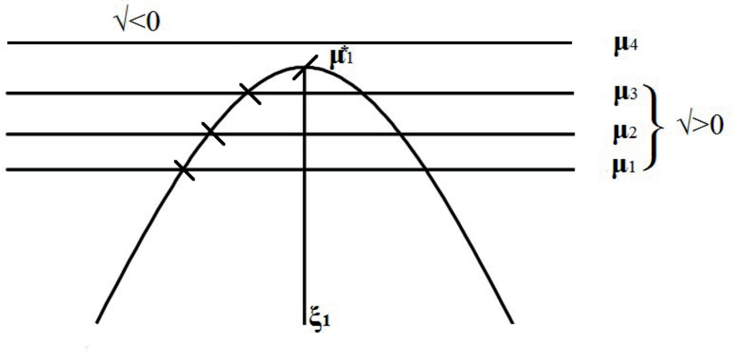

Figure 3 *"Hanging" extremum*

The iterative process of one-sided approximation to the nearest root of a power-law equation is based on local approximation of this equation by a hyperbola or an ellipse, depending on the sign of the curvature at the tangency point. A sufficient condition for one-sided convergence is that all the roots of the polynomial are real. The convergence of iterations to the nearest simple root of the power equation is asymptotically cubic.

If one moves along the equilibrium curve in sufficiently small steps along the load, then jumping over the limit point generates a "hanging" extremum (Figure 3). This will be indicated by the negative sign of the root expression in the denominator in Equation (35).

The solution of t equation (34) of the fourth order gives the desired negative root (closest to zero at $\mu_1 = 0.28$) equal to $\xi_1 = -0.21896$. The second coordinate of the equilibrium point is $\xi_2 = 0.10986$, according to Equation (29). However, the obtained equilibrium point is not the limiting one. Next, the value of μ_1 is increased to 0.3. A similar solution, using Equation (35), gives the equilibrium coordinates $\xi_1 = -0.2422$, $\xi_2 = 0.1399$. With the next value, equal to $\mu_1 = 0.325$, one obtains a negative root expression in the denominator in Equation (35), equal to -1.129·10^{-5}. For the slightly smaller values μ_1 of μ_1, a two-sided estimate: $0.32 < \mu_1^* < 0.325$ is obtained.

The corresponding approximate coordinates of the limit point: $\xi_1^* \simeq -0.3135, \xi_2^* \simeq -0.1998$.

For practical purposes, this is quite sufficient, although this result can be refined by subsequent calculations. The calculated coordinates of the limiting point are close to those shown in Figure 3.3 in the work of Manevich [8].

The relationship between the coefficients of Equations (11), (12) and the types of the wave formation (in the plate or in the ribs) is considered next. Hunt [5], investigating the problem of Tvergaard with a double critical load of the total deflection and wave formation in the plate, established the relationship between the derivatives of potential energy:

$$\Pi_э(\xi_1,\xi_2,\overline{\xi}_1,\overline{\xi}_2) = \frac{1}{6}V_{111}\xi_1^3 + \frac{1}{2}V_{122}\xi_2^2\xi_1 + \\ + \frac{1}{2}\mu_1(V_{11\mu}\xi_1^2 + V_{22\mu}\xi_2^2) + V_{1\bar{\xi}1}\overline{\xi}_1\xi_1 + V_{2\bar{\xi}2}\bar{\xi}_2\xi_2 \quad (36)$$

and the coefficients of the above equations d_i (i=1, 2, 3).

$$\frac{1}{6}V_{111} = -\frac{2}{3}d_1d_3; \frac{1}{2}V_{122} = -2d_2d_3; \frac{1}{2}V_{11} = -d_3, \quad (37)$$

$$\frac{1}{2}V_{22\mu} = -2d_2; V_{1\bar{\xi}1} = 2d_3\mu_1; V_{2\bar{\xi}2} = 4d_2k\mu_1 \,. \quad (38)$$

In order for a double semi-symmetric bifurcation not to be monoclinic, the condition for the positiveness of the root expression T was established in [13]:

$$T = \frac{2V_{11\mu}}{V_{22\mu}} - \frac{V_{111}}{V_{122}} > 0, \text{ or} \quad (39)$$

$$T = \frac{-d_3}{-d_2} - \frac{d_1d_3}{d_2d_3} = \frac{d_3}{d_2} - \frac{d_1}{d_2} > 0 \,. \quad (40)$$

The fulfillment of this condition shows that not one, but three new branches of equilibria in the form of straight lines pass through the double bifurcation point. The wave formation in the plate should be expected in the case of identical signs of the ratios $\frac{d_1}{d_2}$ and $\frac{d_3}{d_2}$. It will occur at the homeoclinic bifurcation point (hyperbolic umbilic). If the signs of the ratios $\frac{d_1}{d_2}$ and $\frac{d_3}{d_2}$ are different, then the wave formation will occur as a result of a local loss of stability of the ribs (anticlinic bifurcation point of the elliptical umbilic). For this reason, it was indicated above that if all the d_i are negative, then the local buckling corresponds to the wave formation in the plate. If $d_1 < 0$ (this is always) and $d_2d_3 > 0$, then the wave formation would occur in the edges of the reinforced plate. According to research of Manevich [8], this feature is also valid for multiple critical loads.

Note that signs of coefficients of Equations (11) and (12) significantly affect the interaction of the general deflection with the local wave formation. If this deflection was positive (all the $d_i < 0$), then its development provokes the wave formation in the plate, since it would receive additional compression.

On the contrary, the ribs in this case will be somewhat relieved by tensile stresses. If the total deflection is negative $(d_1 < 0, d_2 > 0, d_3 > 0)$ then, on the contrary, the plate will be unloaded and the ribs will be loaded with additional compression. This would cause the rippling in the ribs. The described interaction of the general deflection and the effects of

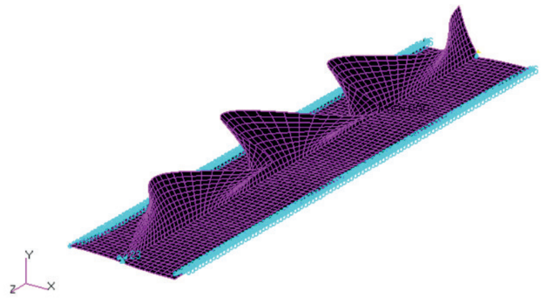

Figure 4 *First custom form*

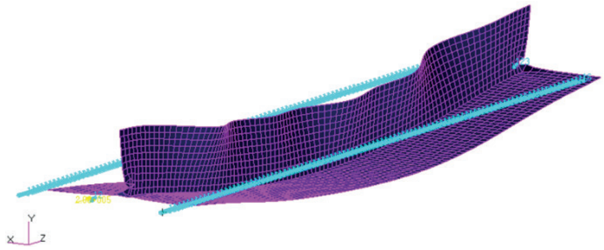

Figure 5 *Deformed equilibrium (at* $\overline{\xi_1} = 0$ *)*

local wave formation is the main feature in behavior of a compressed reinforced plate.

When designing reinforced plates, the appropriate geometric parameters should be chosen so as to exclude the nonlinear interactions between the buckling shapes. For this, it is necessary that the critical loads of the wave formation are significantly higher than the critical loads of the general buckling. This problem requires a separate thorough investigation.

3 Numerical analysis of a T-shaped fragment of a reinforced plate

Algorithms for calculating the critical loads for a reinforced plate, obtaining singular points on equilibrium curves were considered earlier and the relationship between the signs of the coefficients of nonlinear equations and the type of possible wave formation was indicated. By limiting the consideration to one regular T-shaped fragment, one can draw conclusions about the nonlinear behavior of the entire structure containing their totality. A deep understanding of the stability problem, in which there is a question of the interaction of forms, is fully disclosed using the finite element modeling and numerical solutions, taking into account geometric nonlinearity.

In this part of the work, an analysis of the numerical solution using the MSC Software Patran - Nastran software package is presented. Finite elements of a shell type (3131 elements) are selected. The material was considered as absolutely elastic (E=196133.002MPa, μ = 0.3). Boundary conditions: hinged support along the short sides, along the long sides - floating terminations. The load is applied centrally. Geometric parameters were obtained from the relations of Manevich [8, Ch. III] for solving this class of problems h = 0.01 m, b = 0.25 m, t_1 = 0.005 m, b_1 = 0.1 m, L = 1.25 m. The results obtained are compared to the previously obtained results of the semi-analytical solution. The interaction of forms can be presented in two ways. The first option is the interaction of the general and local forms of plate buckling. It manifests itself when the deflection has a positive value. This option does not imply the possibility of the wave formation in the ribs, but only in the plate. The second option is the interaction of the general deflection with the local waveform of the ribs. This deformation is possible only if the deflection of a negative sign develops. The study of this behavior of the structure is reduced to solving the bifurcation problem of stability of a compressed T-shaped fragment. To take into account the local shape in modeling, the shell-type elements were used, which made it possible to consider this model as a set of plate-strips and take into account not only the membrane deformations of the plate, but the transverse deformations of the rib as well. When solving the linear stability problem, the calculation of critical loads for the first ten eigenforms was obtained. It is important to note that the first eight forms correspond to local forms of the wave formation, and only the ninth - to the form of the plate buckling, like an Eulerian rod $(\mathrm{E} = 196133.002\mathrm{Mpa}, \mu = 0.3)$. The first form has seven half-waves, which coincides with the result obtained in [11]. Figure 4 shows the first Eigenform from the linear solution.

Calculating the Euler's critical load, one gets the value $\mathrm{P_{cr}} = 2245803\mathrm{N}$. The result, calculated by the formula, gives an overestimated value of the critical load, since the scheme for calculating the plate as an Euler rack does not take into account the deformation of the plate elements of the bar. Research in a geometrically nonlinear setting without taking into account the initial imperfections $(\overline{\xi_1} = 0)$ made it possible to obtain a point of unstable bifurcation on the equilibrium curve, corresponding to the shape of the wave formation of the edge. The deformed equilibrium and equilibrium curve are shown in Figures 5 and 6, respectively. The critical load value was: $\mathrm{P_{cr}} = 1288934.01\mathrm{N}$.

Figure 5 shows that additional compressive stresses act in the zones adjacent to the free edge of the rib and cause the wave formation bifurcation at loads lower than those obtained from the linear calculation. This deformed state corresponds to equilibrium at the point of unstable bifurcation. The new branch, after passing the singular point, is falling and unstable.

In cases of the non-zero values of the initial imperfections in the total deflection, to construct the bifurcation curve (Figure 7), one sets the amplitudes of the initial deflections and calculates the critical loads. The results obtained $\left(\mu_1 = \dfrac{\lambda}{\lambda_1}\right)$ are in fairly good agreement with the results of the semi-analytical solution (Figure 2).

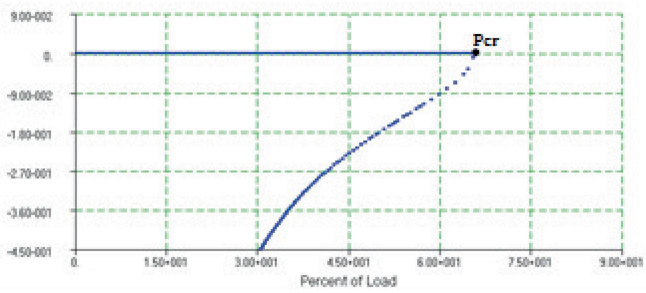

Figure 6 *Equilibrium curve (at* $\overline{\xi_1} = 0$)

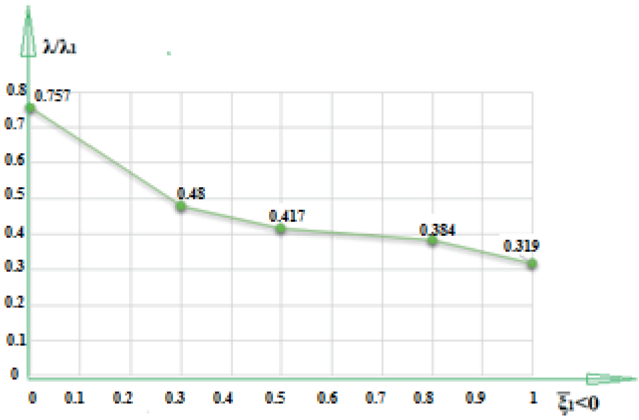

Figure 7 *Bifurcation curve*

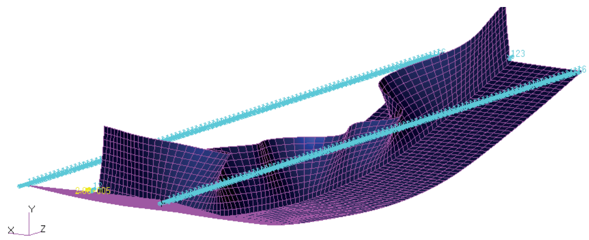

Figure 8 *Deformed equilibrium (at* $\overline{\xi_1} = -0.005\,m$)

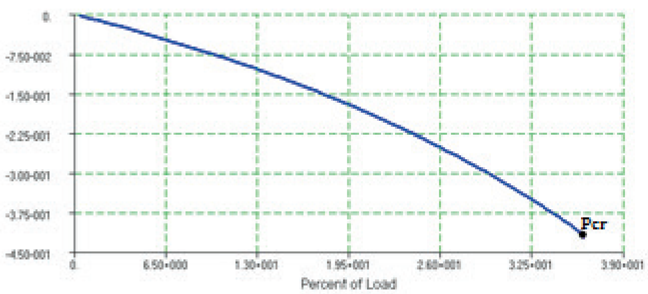

Figure 9 *Equilibrium curve (at* $\overline{\xi_1} = -0.005\,m$)

By setting the amplitude of the initial deflection $\overline{\xi_1} = 0.005$ m, one finds the critical load of wave formation $P_{cr} = 710026.8N$, and the value $\mu_1 = \dfrac{\lambda}{\lambda_1} = 0.417$, which is slightly less than $\mu_1^{bif} = 0.428$, obtained from the semi-analytical solution. The amount of deflection was $\xi_1 = -0.00427$ m, while using Equation (26) one got $\xi_1 = -0.003728$ m. The equilibrium curve and the initial deformed post-bifurcation equilibrium are shown in Figures 8 and 9, respectively.

The amplitudes of the rib deflections from the vertical plane are much larger in comparison to the similar deviations obtained in the problem without initial imperfection.

The load P_{cr} is maximum in the case when the initial imperfection is set in the total deflection $(\overline{\xi_1} = -0.5)$.

To get the limiting point on the equilibrium curve, it is necessary to specify imperfections in the local and general buckling shape. Setting the amplitudes $\overline{\xi_1} = 0.005$ m

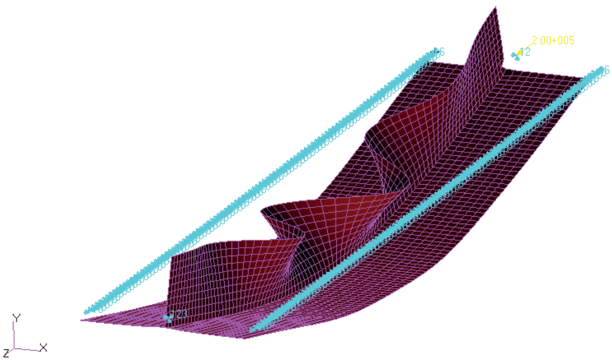

Figure 10 *Deformed equilibrium at the limit point (at* $\overline{\xi_1} = 0.005\,m,\ \overline{\xi_1} = 0.001\,m$)

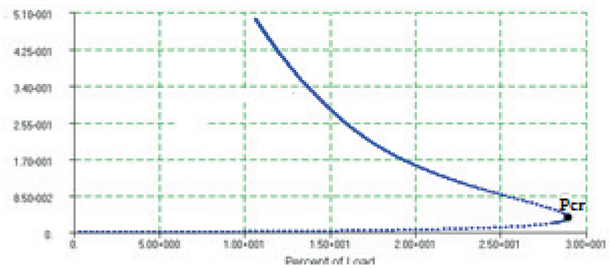

Figure 11 *Equilibrium curve (at* $\overline{\xi_1} = 0.005\,m,\ \overline{\xi_1} = 0.001\,m$)

and $\overline{\xi_1} = 0.001$ m, one finds the value of the critical load $P_{cr} = 567021.126N$ N, then $\mu_1^* = 0.33$, which quite well coincides with the upper estimate obtained from the semi-analytical solution $(0.32 < \mu_1^* < 0.325)$. The total deflection left: $\xi_1 = -0.00673$ m. The deformed balance is shown in Figure 10. The equilibrium curve is shown in Figure 11.

4 Conclusions

The article considers the solution of the nonlinear problem of the reinforced plate stability in the case of multiple and non-multiple critical loads. The most dangerous interaction of shapes is the mutual influence of local buckling of rectangular ribs with the shape of the general deflection. A detailed solution to this problem for the reinforced plates is given in [7]. The reduction in bearing capacity reached 60%. The results of the semi-analytical and numerical solution of stability of the T-shaped fragment of the plate turned out to be quite close. The above equations are suitable for determining the coordinates of singular points on the curves of equilibrium and critical loads.

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