DETECTION OF TRANSPORTATION NETWORK ELEMENTS CRITICAL TO EMERGENCY SERVICE SYSTEMS

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Resume

A study of the vulnerability of emergency service systems leads to the problem of detecting elements (vertices and edges) of transportation networks that are critical to the functionality of these systems. In this paper, methods for identifying the most critical edges and vertices in networks with respect to emergency service systems are presented, and these methods were tested on real data. The primary results of this work are the development and testing of two algorithms that compute a measure referred to as change in transportation performance for edges and vertices in the network. A method for identifying the edges and vertices that are the most critical to the designed emergency service system is presented. Experiments on the transportation network of the Zilina region highlight the importance of probabilistic models of travel time elongation, based on appropriate probability density functions.

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1 Introduction

In practice, many cases are known, where an increase in ambulance driving time on certain road segments or within municipalities significantly affects the functionality of the entire emergency service system. This paper is focused on identifying elements (edges and vertices) of transportation networks that have the greatest impact on the performance of a designed emergency system. For this reason, a metric called transportation performance was introduced by Janacek and Kvet in [1]. Subsequent studies [2], [3] and [4] explored its properties and proposed a function called "change of transportation performance", which quantifies the impact of increased travel times on the designed emergency service system. In [5] and [6], it was shown how a single value can be assigned to any edge in the network using the change of transportation performance. This evaluation enables identification of edges that are the most critical for the designed emergency service system. In this paper, these ideas are extended by experimentally comparing probabilistic and non-probabilistic approaches. Furthermore, a similar method for detecting vertices that are the most critical vertices for the emergency service system was developed. This topic was introduced in [7].

Transportation network G is represented by an ordered quadruple $G = (V,E,w,t)$, where V is the set of vertices (municipalities or crossroads), E is the set of edges (road segments), $w(u)$ denotes the weight of vertex $u \in V$ (e.g., number of inhabitants), and $t(e)$ represents the driving time through edge $e \in E$ (measured in minutes). Let the subset $U \subseteq V$ denote the set of service system customers, and the subset $S \subset V$ the set of located service centers. The transportation performance, as defined in [1], is given by

$$W = \sum_{u \in V} w(u) \cdot d_G(u,S), \tag{1}$$

where $d_G(u,S)$ is the driving time between vertex u and the nearest vertex in S. Now, let the edge $e \in E$ be given, and let x denote the increase in driving time on edge e. The transportation performance with this extension is denoted as $W_e(x)$. The change of transportation performance is the function of one variable:

Table 1 *Distances in networks G and G-e*

i	1	2	3	4
$d_G(u_i,v)$	3	3	1	1
$d_{G\text{-}e}(u_i,v)$	5	4	3	1

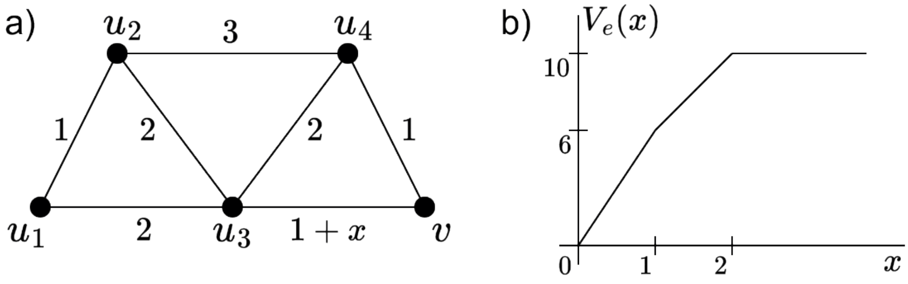

Figure 1 *a) Network G from Example 1, b) Graph of the change of transportation performance*

$$V_e(x) = W_e(x) - W. \tag{2}$$

It has been proven in [2] and [3] that $V_e(x)$ is a piece-wise linear, concave, and non-decreasing function defined on the domain $\langle 0, \infty)$.

Example 1. Consider the network G shown in Figure 1a. The vertex weights are given as $w(u_1) = 3$, $w(u_2) = 2$, $w(u_3) = 1$, and $w(u_4) = 4$. The set S contains only the vertex v, and the affected edge is $e = \{u_3, v\}$. Algorithms for computing the formula of the function $V_e(x)$ are provided in [1] and [5].

1) Compute shortest paths from u_i ($i = 1,2,3,4$) to v in networks G and G-e. Values are given in the Table 1.
2) $W = \sum_{i=1}^{4} w(u_i) \cdot d_G(u_i, v) = 3 \cdot 3 + 2 \cdot 3 + {} + 1 \cdot 1 + 1 \cdot 1 = 18.$
3) $h_i = d_{G\text{-}e}(u_i,v) - d_G(u_i,v)$, where $i = 1,2,3,4$.
4) For every vertex $d_{G\text{-}e}(u_i,v)$, it holds $W_e(x,u_i) = \begin{cases} w(u_i) \cdot (d_G(u_i,v) + x); \text{if } x \in \langle 0, h_i) \\ w(u_i) \cdot d_{G-e}(u_i,v); \text{if } x \in \langle h_i, \infty) \end{cases}.$

Hence $W_e(x) = \sum_{i=1}^{4} W_e(x, u_i)$ and $V_e(x) = W_e(x) - W$. The function $V_e(x)$ is defined as

$$V_e(x) = \begin{cases} 6x & ; x \in \langle 0, 1); \\ 4x + 2 & ; x \in \langle 1, 2); \\ 10 & ; x \in \langle 2, \infty). \end{cases} \tag{3}$$

The graph of this function is shown in Figure 1b.

2 Methods for evaluating edges and vertices

In this section, methods for evaluating and detecting the most critical edges and vertices are developed.

2.1 Evaluation of edges

When the function $V_e(x)$ is defined, one can assign a value to edge e as follows:

1) (Deterministic evaluation.) In the first approach, the area under the function $V_e(x)$ is assigned to each edge e:

$$C_{S,1}(e) = \int_0^T V_e(x) dx, \tag{4}$$

where T is an appropriate upper bound. In tests, the value $T = 60$ minutes is used, because observations in [8] indicate that longer ambulance travel times are rare. Value $C_{S,1}$ represents person-hours (or person-minutes), defined as the number of affected persons multiplied by the delay duration.

2) (Probabilistic evaluation.) If one accounts for the fact that different delay values occur with different probabilities, to each edge is assigned the value:

$$C_{S,2}(e) = \int_0^\infty V_e(x) f(x) dx, \tag{5}$$

where $f(x)$ is a suitable probability density function. In experiments, the Gamma distribution is used [9], since its special case - Erlang distribution - is commonly used to model waiting times in Poisson processes (which, for example, are applied in modelling the occurrence of unexpected incidents in traffic). The probability density function of Gamma distribution is

$$f(x) = \frac{\lambda^\alpha \cdot x^{\alpha-1} \cdot e^{-\lambda x}}{\Gamma(\alpha)}, \tag{6}$$

where $\lambda > 0$ and $\alpha > 0$ are parameters of the distribution and $\Gamma(\alpha)$ is the Gamma function. Value $C_{S,2}$ represents "expected value" of the function V_e.

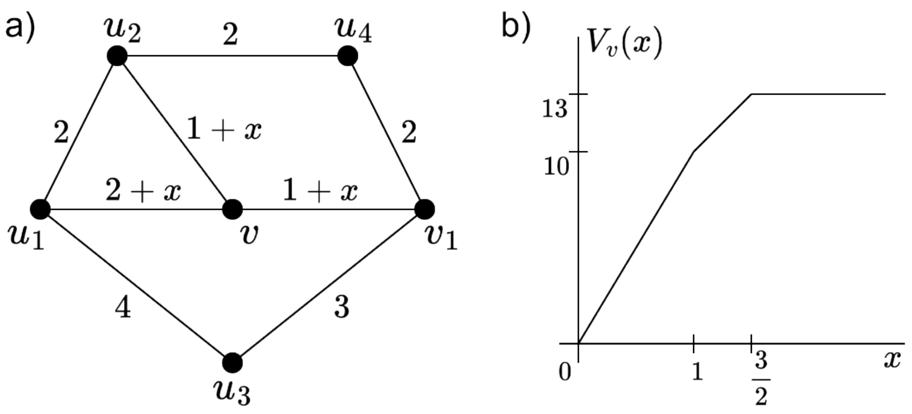

Figure 2 *a) Network G from Example 2, b) Graph of the change of transportation performance*

2.2 Transportation performance of vertices

In this subsection, a function called the change in transportation performance for a vertex v is described. As in the previous section, one starts from the definition of the transportation performance W.

Suppose that, due to unusual circumstances, the driving time through a vertex $v \in V$ is extended. If v is an end-vertex - that is, either a service center $(v \in S)$ or a served customer $(v \in U)$ - then the elongation is x. It can be modelled as an increase in travel time on the edge corresponding to departure from or arrival at the vertex. If v is not an end-vertex, it is assumed that the elongation is $2x$ and it can be modelled as increased travel time on the two edges incident to vertex v, corresponding to arrival and departure. The transportation performance under this elongation at vertex v is denoted by $W_v(x)$. The corresponding change in transportation performance is then given by:

$$V_v(x) = W_v(x) - W. \tag{7}$$

The assumption of elongation time $2x$ for non-end-vertices allows to work with the function $V_v(x)$ in the same way as with the function $V_Y(x_1, \dots, x_k)$, which expresses the change in transportation performance when a set of edges $Y = \{e_1, \dots, e_k\}$ is affected and $x_1, \dots, x_k$ represent the individual driving time extensions on these edges.

In [3], has been proven that the function $V_Y(x_1, \dots, x_k)$ is continuous, piecewise linear, and concave in k variables. In [5], an algorithm for computing explicit formulas for $V_Y(x_1)$ and $V_Y(x_1, x_2)$ is presented. For $k > 2$ affected edges, the computation becomes more complex, and only a few cases admit efficient algorithms. One such case arises when the affected edges $e_1, \dots, e_k$ are precisely those that are incident to a vertex v. In this scenario, assuming $x_1 = x_2 = \dots = x_k = x$, leads to the simplification:

$$V_Y(x, \dots, x) = V_Y(x) = V_v(x). \tag{8}$$

Example 2. Let the network G be as shown in Figure 2a. Customers are represented by vertices u_1, u_2, u_3 and u_4, with weights $w(u_1) = 3$, $w(u_2) = 2$, and $w(u_3) = w(u_4) = 1$. respectively. The service center is located at vertex $v_1 (S = \{v_1\})$, and the affected vertex is v. The graph of the function $V_v(x)$ is shown in Figure 2b. The formula for $V_v(x)$ is

$$V_v(x) = \begin{cases} 10x & ; \textit{if } x \in \langle 0, 1); \\ 6x + 4 & ; \textit{if } x \in \langle 1, 3/2); \\ 13 & ; \textit{if } x \in \langle 3/2, \infty). \end{cases} \tag{9}$$

2.3 Vertices - computation of the function $V_v(x)$

In this subsection, an algorithmic method for computing an explicit formula for the function $V_v(x)$ is presented.

Algorithm

1. For each customer $u \in U$, compute the values $d_G(u, S)$ and $d_{G-v}(u, S)$.
2. Create a list of vertices $u_1, \dots, u_k \in U$ such that $0 < d_{G-v}(u_i, S) - d_G(u_i, S) < \infty$.
3. For $i = 1, \dots, k$, order the values
$$\frac{d_{G-v}(u_i, S) - d_G(u_i, S)}{2}$$
into a non-decreasing sequence.
4. Remove duplicates to obtain a strictly increasing sequence $x_1, \dots, x_m$.
5. Extend this sequence by adding $x_0 = 0$ and $x_{m+1} = x_m + 1$.
6. For $i = 0, 1, \dots, m+1$, compute the values $y_i = V_v(x_i)$.
7. For each interval $\langle x_i, x_{i+1} \rangle$, determine the parameters a_i and b_i of the linear function $y = a_i x + b_i$.

Computation of the values $y_i = V_v(x_i)$ from Step 6:

1. Increase the travel time through every edge incident

to vertex v by the value x_i, resulting in a modified network denoted G_i.

2. Then, compute:
$$W_v(x_i) = \sum_{u \in U} w(u) \cdot d_{G_i}(u,S)$$
$$V_v(x_i) = W_v(x_i) - W\,.$$

As discussed earlier, for $i = 0,\dots,m$, the function $V_v(x)$ is piecewise linear on each interval $\langle x_i,x_{i+1}\rangle$. Therefore:

$$\forall x \in \langle x_i,x_{i+1}\rangle, V_v(x) = a_i x + b_i\,. \tag{10}$$

From analytical geometry, the coefficients of the linear function $y = a_i x + b_i$ are given by:

$$a_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}, b_i = \frac{y_i \cdot x_{i+1} - x_i \cdot y_{i+1}}{x_{i+1} - x_i}\,. \tag{11}$$

The coefficients a_m and b_m for the interval $\langle x_m,\infty\rangle$ are the same as those for the interval $\langle x_m,x_{m+1}\rangle$.

This algorithm provides a systematic method for computing the piecewise-linear function $V_v(x)$.

Example 3 This algorithm is applied to the network and values from Example 2:

1) $d_G(u_1,S) = 3, d_{G\text{-}v}(u_1,S) = 6; d_G(u_2,S) = 2, d_{G\text{-}v}(u_2,S) = 4; d_G(u_3,S) = 3, d_{G\text{-}v}(u_3,S) = 3$ and $d_G(u_4,S) = 2, d_{G\text{-}v}(u_4,S) = 2$.
2) $L = u_1, u_2$.
3) $S_{n\text{–}d} = 1,3/2$.
4) $S_{in} = x_1, x_2$, where $x_1 = 1, x_2 = 3/2$.
5) $x_0 = 0$, and $x_3 = x_2 + 1 = 5/2$.
6) $y_0 = V_v(0) = 0, y_1 = V_v(1) = 10, y_2 = V_v(3/2) = 13$ and $y_3 = V_v(5/2) = 13$.
7) $a_0 = (10 - 0)/(1 - 0) = 10$ and $b_0 = (0 \cdot 1 - 0 \cdot 10)/(1 - 0) = 0$;
$a_1 = (13 - 10)/(3/2 - 1) = 6$ and $b_1 = (10 \cdot 3/2 - 1 \cdot 13)/(3/2 - 1) = 4$;
$a_2 = (13 - 13)/(5/2 - 3/2) = 0$ and $b_2 = (13 \cdot 5/2 - 3/2 \cdot 13)/(5/2 - 3/2) = 13$.

One can observe that the obtained coefficients are identical to those in Example 2.

To complete the analysis and identify the most critical vertices, to each vertex are assigned the numerical values called critical values, denoted $C_{S,1}(v)$ and $C_{S,2}(v)$. For computation of the value $C_{S,1}(v)$, one can assign to each vertex v the area under the function $V_v(x)$ (T = 60 minutes):

$$C_{S,1}(v) = \int_0^T V_v(x)dx\,. \tag{12}$$

The computation of $C_{S,2}(v)$ involves a probability density function that models the probability of travel time increases through the vertex (e.g., due to congestion at a crossroad or delays in a municipality):

$$C_{S,2}(v) = \int_0^\infty V_v(x)f(x)dx\,, \tag{13}$$

where $f(x)$ is an appropriate probability density function. In our tests, we again consider the Gamma distribution. The values of $C_{S,1}$ and $C_{S,2}$ can be treated similarly to the corresponding values defined for edges.

3 Computational results

3.1 Edges

In this section are presented the computational results from an experiment in which the values $C_{S,1}$ and $C_{S,2}$ were assigned to the edges of a real transportation network. As a testing benchmark is used the transport network of the Zilina self-governing region in Slovakia. This network consists of 432 vertices, of which 315 represent municipalities with non-zero weights; the remaining vertices represent crossroads. Ambulance stations are located in exactly 29 of these municipalities. The values $C_{S,1}$ and $C_{S,2}$ are computed for all 494 edges in the network. Selected results from the tests are presented in Table 2. All parameters α and λ are model parameters chosen for the testing purposes.

One can observe that the results for the values $C_{S,1}$ and $C_{S,2}$ differ. It is suggested that criterion $C_{S,2}$ is more suitable for identifying the most critical edges, as it also accounts for the probabilities of driving time elongation. The selection of appropriate values for the parameters λ and α is an important issue and some results can be seen in Table 3.

Table 2 *Top five edges with the highest $C_{S,1}$ and $C_{S,2}$ values in the Zilina self-governing region (for parameters $\lambda = 2$ and $\alpha = 4$)*

$C_{S,1}$			$C_{S,2}$		
edge		value	edge		value
Trstena	Liesek	129 600	Zilina	crossr. 353	220
Bytca	crossr. 349	95 400	crossr. 411	Turcianske Teplice	214
crossr. 349	Hvozdnica	95 400	Martin	Kostany nad Turcom	200
crossr. 411	Turcianske Teplice	88 859	Trstena	crossr. 346	199
crossr. 327	Turzovka	81 000	Tvrdosin	crossr. 346	184

Table 3 *Top five edges for parameters $\lambda = 0.5$ and $\alpha = 4$, or $\lambda = 2$ and $\alpha = 8$ respectively*

Parameters	$\lambda = 0.5$, $\alpha = 4$			$\lambda = 2$, $\alpha = 8$	
edge		value	edge		value
crossr. 411	Turcianske Teplice	780	crossr. 411	Turcianske Teplice	422
Martin	Kostany nad Turcom	616	Zilina	crossr. 353	377
Trstena	Liesek	576	Martin	Kostany nad Turcom	375
crossr. 411	crossr. 412	484	Trstena	crossr. 346	306
Terchova	crossr. 371	475	crossr. 333	Cadca	303

Table 4 *Top five vertices with the highest $C_{S,1}$ and $C_{S,2}$ values in the Zilina self-governing region (for parameters $\alpha = 1$ and $\lambda = 0.5$)*

Vertex	$C_{S,1}$	Vertex	$C_{S,2}$
Cadca	78996	Bytca	1952
Svrcinovec	51004	Cadca	1931
Bytca	32227	Svrcinovec	1906
Martin	27728	crossr. 353	1753
crossr. 411	23806	Martin	1458

Since the results vary across different parameter sets, the following methods for deriving an optimal set of parameters are proposed.

The optimal scenario arises when the sufficient data on ambulance driving times are available for each road segment in the network. In such cases, it is possible to apply statistical methods and construct an appropriate probabilistic model (with a density function $f(x)$) for every road segment. However, this situation is uncommon. For most road segments, data are limited or unavailable.

In these cases, one can refer to models describing the dependency of driving time on distance for fire engines and ambulances, as proposed in [10] and [11]. These works describe how the median, mean, and coefficient of variation of driving time depend on distance. For example, the dependency of the median driving time can be expressed as follows:

$$m(d) = \begin{cases} c\sqrt{d}, d \leq d_0 \\ ad + b, d > d_0 \end{cases}, \tag{14}$$

where a, b, c, and d_0 are parameters derived from velocity, acceleration, and distance.

The relationships between the parameters λ and α of the gamma distribution and its mean, median, and variance are well established [12]:

$$\begin{aligned} &mean = \frac{\alpha}{\lambda}, median \approx \frac{\alpha}{\lambda} \cdot \left(1 - \frac{1}{3\alpha + 0.2}\right), \\ &variance = \frac{\alpha}{\lambda^2}. \end{aligned} \tag{15}$$

Given estimates of the mean, median, or variance, one can derive the parameters λ and α for any road segment of length d. These computations are prepared for all self-governing regions in Slovakia.

3.2 Vertices

Similarly to the case of edges, data from the Zilina Region are used for testing. The values $C_{S,1}$ and $C_{S,2}$ are assigned to the vertices. Some of the results are presented in Table 4.

It should be emphasized that the experiments in this section were conducted solely for the purpose of testing the algorithm itself. When pursuing the practical results, it would be necessary to distinguish between the intersections and municipalities, as their traversal times will likely need to be represented by different models. Identifying such models will be the subject of future research.

4 Conclusion

This study was focused on identification of edges and vertices in transportation networks which are critical for designed emergency service systems. The detection of critical edges is built on our previous research. In this paper, are presented the experimental results demonstrating that a probabilistic approach is essential for evaluating the impact of driving time elongation on emergency service performance.

The problem of identifying critical vertices is also presented and a method for calculating changes in transportation performance for any given vertex is proposed. However, evaluating vertices - similar to the evaluation of edges - using an appropriate probabilistic function remains an open problem.

To assign each vertex v a value such as:

$$C_{S,2}(v) = \int_0^\infty V_v(x) \cdot f(x)\, dx\,, \tag{16}$$

an appropriate probability density function *f*(*x*) is required. It is supposed that ideas developed by [8], [11], and [13] could provide the necessary tools to complete the evaluation of critical vertices.

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Conflicts of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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