CONSTRUCTION OF OPTIMAL TRANSITION CURVES ON RAILWAYS BASED ON CURVATURE

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Resume

Parametric transition curves designed to guarantee a smooth transition from tangent (straight) tracks to circular segments in the horizontal-plane projection of a railway track alignment. A rigorous theoretical framework was developed to express the functional dependence of curvature, deflection angle, and Cartesian coordinates on the curve's arc length. To validate the theoretical derivations, a detailed computational experiment was performed, fully confirming their correctness and accuracy. Furthermore, a methodology for constructing coordinate tables as a function of arc length - highly valuable for practical railway design and field layout - is provided. A robust mathematical apparatus for the curvature-based design of railway transition curves is established. The results demonstrate that the coordinates of the circular arc center were optimized and refined to the required level of precision.

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Nomenclature

Latin characters

- a : Curvature parameter / Maximum value of curvature
- C_1, C_2: Constants of integration
- F : Elliptic integral of the first kind
- F^{-1}: Inverse function of the elliptic integral
- k : Curvature of the transition curve
- L: Total length of the transition curve, m
- p: Scaling coefficient of the argument
- R: Radius of the circular track section, m
- s: Arc length of the curve, m
- V: Varied vector for optimization $(|V_0,V_1|)$
- x,y: Cartesian coordinates of the curve

Greek characters

- ρ : Radius of curvature $(\rho = 1/k)$
- φ : Turning / deflection angle of the curve, °
- Subscripts
- x_A, y_A : Parameters evaluated at the starting (initial) point of the transition curve
- x_B, y_B: Parameters evaluated at the endpoint (junction with circular arc) of the transition curve
- x_C, y_C: Parameters related to the center of the circular track section

1 Introduction

The railway infrastructure design is an intricate, multi-stage engineering process. A critical component of this workflow is the configuration of transition curves that connect straight track sections with circular arcs. In the early periods of railway engineering, horizontal curves were composed of basic geometric elements, namely straight lines and simple circular arcs. However, driven by modern technological advancements and increasingly stringent requirements for passenger comfort and vehicle safety, developing more advanced and dynamically optimal curve geometries has become indispensable.

The straight track sections ensure more stable train motion, since they are not subjected to lateral forces caused by centrifugal action. However, when a train moves along a circular arc, it begins to experience a centrifugal force inversely proportional to the radius. This results in the emergence of lateral loads, which cause passenger discomfort and accelerate the wear of railway equipment. In addition, dynamic shocks occur when passing from a straight section to a curved one. To eliminate these shortcomings, various types of mathematical curves have been introduced into the design of transition curves. Modern methods for designing transition sections include the use of computer-aided design (CAD) systems and the finite element method (FEA). In Marco Guerrieri's book *Fundamentals of Railway Design*, the main aspects of railway line design for both conventional and high-speed railways are considered [1]. The clothoid is widely used as a transition curve. However, the law governing the change of curvature of the clothoid along the track is regarded as unsafe under real operating conditions, especially at variable speeds near the beginning of the transition curve. Even extending the clothoid indefinitely does not fully resolve this problem [2]. To overcome this drawback, more advanced curves should be employed. First of all, such curves should have a nonlinear curvature law and should make it possible to increase the length of the transition curve to the required extent. Lobanov et al. noted that, in road design, the deficiencies of transition curves constructed by traditional methods cannot be eliminated even by replacing them with splines, since not every spline can be regarded as a transition curve. This leads to the conclusion that the problem should be approached not as the construction of a purely geometric line, but as the development of a functionally substantiated path trajectory. Such an approach makes it possible to ensure comfortable and safe motion with variable speed on curved sections [3]. In [4], the authors recommend the use of the *Symmetrically Projected Transition Curve* (SPTC), obtained from a comparison of clothoid and cubic curves. However, a large number of approximate relations were used in deriving the SPTC expression. In [5], the application of cubic curves to railways was investigated. Zboinski and Wojnica proposed a polynomial-curve method using 9th- and 11th-degree polynomials and claimed that this improves passenger comfort on long transition sections (over 150 m) [6]. Basak [7] touched on the application of different types of curves (clothoid curve, cubic curve and Bernoulli leminscate) in the transition curve and tried to justify their application. In [8], the authors investigated the influence of the shape of curvilinear sections in a high-speed railway track plan on traffic dynamics. It was determined that the best dynamic performance is achieved in the curved section of the alignment equipped with transition curves whose curvature function is represented by a 5th-degree polynomial. In [9], the objective was to be able to recreate (i.e., model) the existing geometric layout of roads consisting of compound curves, so that it is then possible to correct this layout. An analytical method for designing track geometric systems was used, adapted to the mobile satellite measurement technique, in which calculations are carried out in the appropriate local Cartesian coordinate system. The obtained possibilities of modeling the compound curves are illustrated by the provided calculation example. Many factors must be considered in the design of high-speed railways. These conditions pertain to track design, traffic organization, the solution of technical problems, etc. [10]. In [11], based on the analysis of statistical data on railway accidents, it was concluded that one of the causes of incidents is track curvature. A mathematical model for automatic speed control depending on the track curvature, the presence of obstacles, and track defects. The aim of the research presented was to develop a method for modifying compound curves in railways, i.e., geometric systems composed of two (or more) circular arcs with different radii, directed in the same direction and directly connected to each other. As a result of the conducted research, the transition curve, determined with strict curvature conditions, was evaluated as the most advantageous option. [12]. in the study conducted in [13] was investigated the use of 9th and 11th-degree polynomials for the optimization of railway transition curves. In sections longer than 150 meters, these higher-degree models increase passenger comfort and minimize rail wear by reducing lateral acceleration. The impact of curvature smoothness at the extreme points of railway transition curves on vehicle dynamics is investigated in the study conducted by Zboinski and Chrostowski. The research demonstrates that for long curves typical of high-speed lines, optimizing for smooth end-points reduces lateral forces and minimizes rail wear [14]. In the article presented by Kobryn, a design method of vertical arcs using the so-called general transition curves is described. These curves describe the entire curvilinear transition between two straight lines using a single equation [15].

To ensure smooth dynamic transitions at the junction points and along the transition curves, it is essential for the first, second, and third-order derivatives to change continuously. To achieve this, the mathematical law defining the curvature must ensure that the transition at the boundaries is seamless and mathematically smooth. Depending on the boundary conditions and geometric constraints of the railway track axis, the sinusoidal curvature expression can yield either a strictly monotonic profile or a highly smooth non-monotonic profile (a curvature hump). These include ensuring the starting point of the transition curve coincides with the end of the straight track section and its endpoint coincides with the start of the circular curve, as well as maintaining the equality of tangents and curvatures at these points. Taking these factors into account, the curvature in the presented article is assumed to follow a sinusoidal dependence. The differential equation of curvature in Equation (1) was solved here for the first time as a special case in the Cartesian coordinate system to derive Equation (3), which represents the transition curve [16]. Equation (3) was obtained by solving the inverse problem of the curvature differential equation. The curvature of this line varies within the interval $[0,1/R]$, meaning that it can merge smoothly with both straight and curved track elements. Since Equation (3) meets the high requirements set for the line curvature, it has been utilized in solving various engineering problems, including the construction of transition curves for roads and railways [17-18].

Extensive literature reviews indicate that an explicit expression, derived from the exact analytical solution of the differential equation of curvature, has not been previously applied to the design of railway transition curves. Utilizing a curvature-based formulation inherently guarantees the smooth and continuous variation of curvature along the entire transition trajectory from the outset. Consequently, this approach significantly reduces the number of design iterations required to establish the final alignment. Moreover, the boundary conditions strictly ensure that the curvature and deflection angles at the extreme points of the transition curve match those of the adjacent tangent tracks and circular segments precisely.

2 Materials and methods

It is well known that, for transition curves, the curvature is defined as $k(1/\rho)$, where the variable ρ (the radius of curvature) changes monotonically from $\rho = \infty$ at the beginning of the curve to $\rho = R$ at its end, with R being the radius of the circular track section. The radius of the circular section is selected based on of satisfying specific technical requirements. If the curvature function of the transition curve, $k(1/\rho)$, is known in advance, then its connection with the other track sections without discontinuities can be achieved more easily and accurately. This can be accomplished by analytically solving the differential equation of curvature, which may be regarded as an inverse problem. In the present article, for the construction of the transition curve, an expression obtained from the analytical solution of the differential equation of curvature in Equation (2) in the Cartesian coordinate system, as a particular case, is proposed. The differential equation of curvature [19]:

$$k(x) = \frac{\dfrac{d^2y}{dx^2}}{\sqrt{\left(1 + \left(\dfrac{dy}{dx}\right)^2\right)^3}}, \tag{1}$$

was solved under the assumption that.

$$k(x) = -a\sin(px) \tag{2}$$

One of the main reasons for using a sinusoidal curve was to ensure a smooth variation of curvature up to the beginning of the circular arc.

As a result, the following equation was obtained:

$$y(x) = \frac{1}{p}\ln\left(\sin(px) + \sqrt{B^2 - \cos^2(px)}\right) + C_1, \tag{3}$$

where:
C_1 - is an integration constant. Here, $B=p/a$;
a - is the maximum value of curvature,
p - is the scaling coefficient of the argument.

In Equation (3), the coordinates are given in the Cartesian coordinate system. In the design of railway transition curves, the coordinates are mainly used as functions of the arc length (parametrically $x(s)$, $y(s)$). To obtain the formula expressing the dependence of the coordinates on the arc length, the following transformations were applied.

As is well known, the length of a curve s is determined by the following expression [20]

$$s = \int \sqrt{1 + (y')^2}\, dx. \tag{4}$$

If y' is calculated from Equation (3) and substituted into Equation (4), then

$$\sqrt{1 + (y')^2} = \frac{p}{\sqrt{p^2 - a^2\cos^2(px)}}. \tag{5}$$

Then, one obtains:

$$s = \int \frac{p}{\sqrt{p^2 - a^2\cos^2(px)}} dx. \tag{6}$$

This integral is an elliptic integral and can be written in the form:

$$s = \frac{1}{p}F\left(px, \frac{a}{p}\right), \tag{7}$$

here F is the elliptic integral of the first kind [21].

After that, the dependence of the curvature on the curve length is determined. For this purpose, the dependence of x on s is obtained from Equation (7):

$$px = F^{-1}\left(ps, \frac{a}{p}\right), \tag{8}$$

where F^{-1} is the inverse function of the elliptic integral. If the last formula in Equation (3) is taken into account, one obtains

$$k(s) = -a \cdot \sin\left(F^{-1}\left(ps, \frac{a}{p}\right)\right). \tag{9}$$

Equation (9) is simplified for calculation. As is well known, the curvature is:

$$k = \frac{d\varphi}{ds}. \tag{10}$$

On the other hand, since s is expressed in terms of x, and taking into account Equations (2) and (4), the following expression is obtained:

$$\frac{d\varphi}{ds} = k(x)\frac{ds}{dx} = -a\sin(px)\sqrt{1 + (y')^2}. \tag{11}$$

By differentiating both sides of Equation (6) and performing simple transformations, the following equation can be obtained:

$$\frac{ds}{dx} = \frac{p}{\sqrt{p^2 - a^2\cos^2(px)}}. \tag{12}$$

By integrating Equation (12), one obtains Equation

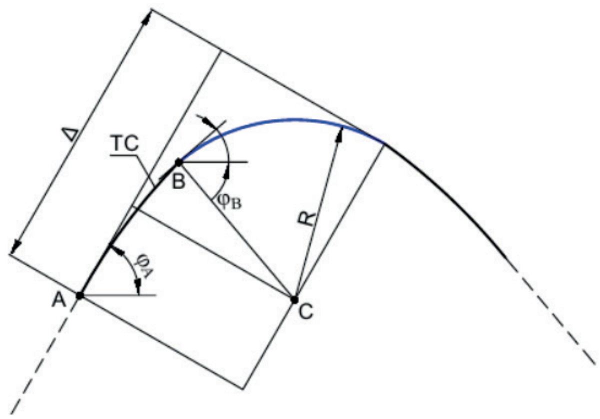

Figure 1 *Scheme of the transition curve*

(13). This is the formula for the length of the transition curve.

$$s = \frac{1}{a}\arcsin\left(\frac{a}{p}\cos(px)\right) + C_2 . \tag{13}$$

To derive the dependence $x(s)$ from Equation (13), the simple transformations are performed:

$$\cos(px) = \frac{p}{a}\sin(as + C_3), \tag{14}$$

where, $C_3 = -aC_2$.

From expression $\sin(px)$ from the expression $\sin^2(px) + \cos^2(px) = 1$, substituting it into Equation (2) and considering Equation (14), one obtains:

$$k(s) = -a\sqrt{1 - \frac{p^2}{a^2}\sin^2(as + C_3)}, \tag{15}$$

Equation (15) is the formula for the dependence of curvature on arc length.

Now is considered the dependence of the angle on the curve length. As is well known, the dependence of the angle on curvature is given by the integral in the form of [15]:

$$\varphi(s) = \int k(s)ds . \tag{16}$$

If Equation (15) is taken into account in Equation (16), one obtains:

$$\varphi(s) = -a\int_0^L \sqrt{1 - \frac{p^2}{a^2}\sin^2(as + C_3)}ds . \tag{17}$$

Equation (17) is the formula for the dependence of the turning angle on the curve length.

The dependence of the coordinates on length is determined by the following expressions [22]:

$$\begin{aligned} x(s) &= \int \cos(\varphi(s))ds \\ y(s) &= \int \sin(\varphi(s))ds \end{aligned} . \tag{18}$$

In Equation (18), the point where the straight track section joins the transition curve is taken as the origin of coordinates. Using Equation (18), it is possible to determine the coordinates of the railway transition curve in the Cartesian coordinate system as functions of the arc length.

3 Results and discussions

In the present article, the modelling of planar transition curves using their parameterization with respect to arc length is proposed. Such an approach assumes that the Cartesian coordinates of the modeled curve are functions of its arc length, i.e., $(x(s), y(s))$. To verify the validity of the obtained expressions, the principal parameters must be satisfied at the junction points (points A and B) (Figure 1). To initiate the calculation, initial values are assigned to the boundary conditions, with the meter taken as the unit of length. For example, one can assume the length of the transition curve to be $L = 40\ m$. The transition curve varies over the interval $s = [0, L]$, where $R = 500\ m$ is the radius of the circular track section, $\varphi_A = 0^\circ$ is the angle formed by the tangent at the initial point of the transition curve with the horizontal line, $\varphi_B = 15^\circ$ is the angle formed by the tangent at the end point of the transition curve with the horizontal line, and $x_A = 0\,m, y_A = 0\,m$ and x_B are the coordinates of the initial and end points, respectively (Figure 1). In Figure 1, TC denotes the transition curve, and point C is the center of the circular track. All the linear dimensions were normalized by dividing them by $R = 500\ m$.

Using the boundary conditions given above, Equations (15), (17) and (18) are solved simultaneously to determine the coordinates of the transition curve, and the constants are thereby obtained. The angles used in Equation (19) are given in radians, and the system is solved in the MathCAD program [23].

$$\begin{cases} \frac{1}{R} = -a\sqrt{1 - \left(\frac{p}{a}\right)^2 \sin^2(aL + C_3)} \\ C_3 = \arcsin\left(\frac{a}{p}\right),\ s = 0,\ k(0) = 0 \\ \varphi_B = -\int_0^L \sqrt{1 - \left(\frac{p}{a}\right)^2 \sin^2(as + C_3)}ds. \end{cases} \tag{19}$$

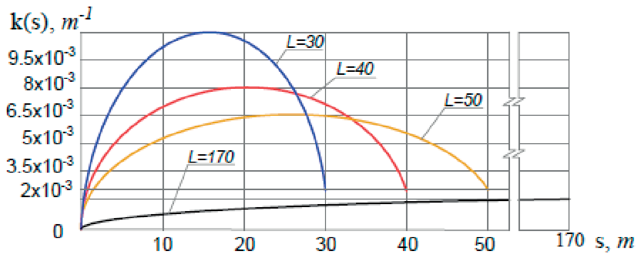

Figure 2 *Variation of curvature along the transition curve length s for different total lengths (L=30 m, L=40 m, L=50 m, 170 m) at a constant circular radius R=500 m and deflection angle* $\varphi_B = 15°$

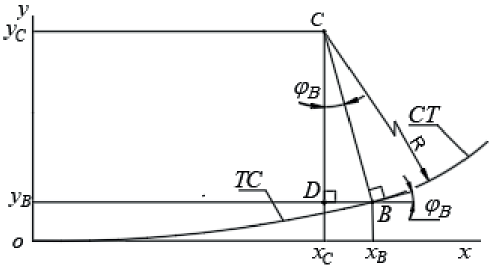

Figure 3 *Coordinate representation of the transition curve*

If the values of the constants, *a. p,* C_3 obtained from Equation (19), are substituted into Equation (15), then $k(0) = 0$ and $k(40) = 0.002\ m^{-1}$ are obtained. A graphical representation of Equation (15) is shown in Figure 2. As can be seen from the Figure 2, the graph of the dependence of curvature on the length of the transition curve is, as noted above, a sinusoidal curvature function evaluated through elliptic integrals (Equation (9)). Equation (15) ensures a smooth transition to the straight line at the initial point of the curve ($k(0) = 0$), and at the end point it provides a smooth transition to a circle of radius 500 ($k(40) = 0.002\ m^{-1}$). Expression (18), which ensures smooth connection with the straight line and the circle of radius 500 m, guarantees the absence of any jumps in train motion along the entire transition curve.

To comprehensively validate the generalized mathematical model and evaluate the geometric boundary integration, Figure 2 illustrates the parametric curvature profiles generated for different transition lengths (L=30 m, L=40 m, L=50 m and L=170 m at a targeted circular curve radius of R = 500 m. The computational curves demonstrate absolute boundary compliance based on the formulated sinusoidal law. At the starting tangent point (s=0), the curvature is exactly zero (*k=0*). As the arc length progresses, the curvature profile exhibits two distinct behaviors depending on the transition length: for shorter lengths (L=30 m, L=40 m and L=50 m), a non-monotonic curvature hump is observed; whereas for the longer length (L=170 m), the curvature increases strictly monotonically. At the exit tangent point, all configurations converge precisely to the exact curvature value of the adjacent circular track segment, $k(L) = \frac{1}{R} = 0.002\,m^{-1}$. The elimination of any geometric boundary mismatch under varying parameters confirms that the optimization loop and boundary constraints are rigorously satisfied, providing a stable, smooth geometric path for vehicle transition.

The graphical representation of Equation (18) is shown in Figure 3. As illustrated in Figures 2 and 3, the curvature profiles evaluated across different transition lengths adapt smoothly along the trajectory (*TC*) and converge precisely with the circular (*CT*) track segment without any geometric discontinuities or alignment kinks.

To determine the coordinates of the center of the circular track at point *C*, the scheme shown in Figure 3 is used. As can be seen from Figure 3, $\angle C = \varphi_B$. Taking into account that $|CB| = R$ from ΔBCD triangle $|CB| = R$, the coordinates are determined by:

$$\begin{aligned} x_c &= x_B - R\sin(\varphi_B) \\ y_c &= y_B - R\cos(\varphi_B), \end{aligned} \tag{20}$$

where $x_B = \int_0^L \cos(\varphi(s))$ and $y_B = \int_0^L \sin(\varphi(s))$ are determined by the corresponding expressions. For the graphical representation of Equation (20) and for plotting the circular track section, the AutoCAD program, which belongs to CAD systems, was used.

The outgoing transition curve can be constructed by symmetrically rotating the incoming transition curve with respect to the center and midpoint of the arc of the circular track section.

Table 1 presents the practically significant dependence of the coordinates of the transition curve on the track length in the Cartesian coordinate system.

To quantitatively and analytically justify the stated advantages of the proposed curve over the traditional clothoid, the higher-order derivatives of both geometric profiles must be evaluated. In a classical clothoid, the curvature varies linearly $(k(s) = C \cdot s)$, which means its first derivative - directly representing the lateral jerk indicator—undergoes an instantaneous step-jump from 0 to C at the initial tangent point (s = 0). This mathematical discontinuity causes dynamic shocks in train motion. Conversely, the proposed sinusoidal-based curvature profile, defined through elliptic integrals, ensures that both the curvature $k(s)$ and its derivative $k(s)'$ smoothly approach zero at the boundaries (s = 0 and s = L). This absolute mathematical continuity eliminates the step-jumps in lateral jerk, theoretically guaranteeing the ‘absence of any jumps in train motion’ and validating the superior smoothness of the layout.

The optimization of the (x_C, y_C) coordinates of the center of the circular track section by the least-squares method was adopted as the objective function,

$$f(V) = \left[\sqrt{(V_0 - x_B)^2 + (V_1 - x_B)^2} - R\right]^2 + \\ + \left[a\tan 2(V_0 - x_B, V_1 - y_B) - \left(\varphi_B + \frac{\pi}{2}\right)\right]^2, \quad (21)$$

where the atan2 function [24] is used in MathCAD to transform Cartesian coordinates into polar coordinates (the angle is measured from the positive direction of the y-axis). Here, the varied vector V (V_0 = x_C, V_1 = y_C) ensures that the center lies on the normal drawn from point B. As can be seen, Equation (21) consists of two terms. The first term ensures that the distance from point B to the center of the circular track section coincides with the radius of the circular section, while the second term ensures the coincidence of the tangent values at that point. The Hooke-Jeeves algorithm [25] is used to find the minimum of Equation (21). By means of this algorithm, the coordinates of the center of the circular track section are refined in an optimal manner. Upon establishing the optimal center of the circular curve, the deflection angle at point B effectively reaches its optimized value of $\varphi_B = 14.99°(\approx 15°)$.

Only after the coordinates have been refined is Table 1 compiled, which is of practical significance and intended for use in setting out the transition curve in the field.

4 Conclusions

In this study, optimal transition curves based on curvature were derived to ensure the smooth transition of the horizontal railway axis projection from straight sections to circular sections or the connection of two circular sections. The results of the research and the suggestions are:

- boundary conditions have been determined for identifying the constants in the expression derived from the differential equation of curvature;
- using the equation of the curve obtained in the

***Table 1** Dependence of transition curve length on curvature and coordinates*

No.	Length of the transition curve s, m	Curvature of the transition curve k, m^{-1}	Coordinates of the transition curve	
			x, m	y,
1	0	0	0	0
2	2	0.003553688	1.999994241	0.003844319
3	4	0.004897650	3.99991003	0.021517357
4	6	0.005836309	5.99955558	0.058650103
5	8	0.006545556	7.998630603	0.119045747
6	10	0.007093860	9.996743294	0.205552391
7	12	0.007515983	11.99342772	0.320356169
8	14	0.007831925	13.98816163	0.465124672
9	16	0.008053865	15.98038465	0.641088111
10	18	0.008189214	17.96951702	0.849088823
11	20	0.008242095	19.95497881	1.089611775
12	22	0.008214043	21.93620982	1.362802620
13	24	0.008104248	23.91269021	1.668476201
14	26	0.007909430	25.88396231	2.006116024
15	28	0.007623296	27.84965373	2.374866069
16	30	0.007235353	29.80950315	2.773509321
17	32	0.006728480	31.76338944	3.200432145
18	34	0.006073602	33.71136752	3.653561763
19	36	0.005216133	35.65371651	4.130256192
20	38	0.004030037	37.5910166	4.627085375
21	40	0.002000000	39.5243152	5.139279766

Cartesian coordinate system, the theoretical foundations of the parametric dependence of the coordinates on the curve length have been developed;
- the validity of the obtained expressions has been verified and confirmed by means of a computational experiment; it has been shown that the motion along the entire track changes smoothly without any discontinuities;
- to ensure smooth continuity, the coordinates of the center of the circular track section have been optimized;
- a practically significant table showing the dependence of the coordinates of the railway transition curve on length and curvature has been compiled;
- by applying the proposed methodology, with the use of computer technologies, the construction of railway transition curves can be automated;
- the parameters presented in the table of the dependence of the coordinates on the transition-curve length can be used directly for setting out the track in the field.

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Conflicts of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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